*Further Researches in the Theory of Quintic Equations.**

BY EMORY McCLINTOCK.

1. This paper comprises in substance four successive parts: first, a preliminary classification of quintics between reducible and irreducible, and again between resolvable and unresolvable (paragraphs 2–6); secondly, a simplified restatement of my earlier discoveries (7–16); thirdly, the presentation of the necessary form of the coefficients of the general resolvable quintic (17–32); and lastly, the development of a theorem according to which any given resolvable quintic engenders another for which my sextic resolvent has the same rational value (33–42). In the course of the first part, a method is presented (4–5) for detecting the rational factors of reducible quintics, a method which is applicable as well to equations of other degrees; and this is followed (6) by a method for recognizing quintics which are unresolvable because of their having two and only two imaginary roots. The second part recalls my paper of 1885 entitled "Analysis of Quintic Equations," which was published in the American Journal of Mathematics, vol. VIII, pp. 45–84.† In that paper I showed that there are three cyclic functions of the roots, functions which have a rational value when the quintic is resolvable, namely, t, v, s, connected by the relation $s = t^2 v$, two of which must be taken into account in any simple discussion of the resolution of the quintic. Regarding the recognition of these quantities as my most important contribution to the development of the subject, I dwelt repeatedly upon their usefulness, and gave particulars of earlier investigations which had failed either of success or of simplicity for want of these necessary auxiliaries.

* Read at the Toronto meeting of the American Mathematical Society, August 17, 1897, when copies of a printed sheet containing the formulæ were supplied, for convenience, to the members present.

† I take this opportunity to point out certain *errata:* p. 47, in (5) and (10), $\frac{1}{16}$ is blurred; p. 51, in (32), for u_3^2 read u_2^3; p. 52, in (34), for y read w; last line of p. 59 and second of p. 60, for $\phi\psi^{-1}$ read $\psi\phi^{-1}$; p. 60, in (47), insert ϕ after (5D); p. 60, line 8 from bottom, for 2 read 3; p. 68, line 15 from bottom, for 16 read -16; p. 73, in (106), for d^6 read d_6; p. 75, line 11 from bottom, insert v^{-2} after p^2; next line, for p^2v^{-2} read p^2v^{-4}; next line, for pv^{-1} read pv^{-2}.

Starting from the well-known theory of Bezout and Euler, who assigned four elements, say u_1, u_2, u_3, u_4, as functions of the roots, I developed two chief auxiliary equations containing only t and v in addition to the coefficients, from which by elimination I produced two sextic resolvents, one in t, which was new, the other in v, which was a simplified reproduction of the only resolvent previously known, that of Malfatti. I also showed how a resolvent in s was readily derivable from that in v, and developed formulæ for determining the other two of the quantities t, v, s, whenever one of them became known by means of a resolvent. Finally, I supplied formulæ for determining the roots of the quintic from ascertained values of t and v. In the second part (7–16) of the present paper, besides simplifying one of the formulæ last mentioned, I reproduce much of the work referred to, but in a different order, which appears to reduce the algebraic labor to a minimum. In fact, I adopt a new method which might be applied to equations of other degrees, and which is the precise reverse of that of Bezout and Euler: instead of defining the elements as functions of the roots, I start with the elements and define the various quantities with which I deal, including the coefficients and the unknown quantity itself, as functions of the elements. In the third part, I develop (17–20) formulæ by which, assigning rational values at will to four parameters, we are enabled to produce the coefficients, and the quantities t and v, for all possible resolvable quintics; I consider (21–25) the modifications of this system which become necessary in critical cases, remark (26) upon the difficulty of constructing resolvable quintics of the form $y^5 + 10\gamma y^3 + 10\delta y^2 + \zeta = 0$, and give reasons (27) why simpler parameters cannot be devised. After remarking (28) that in general there are four conjugate quintics for which t and v have identical values, I refer (29–32) to the history of previous partial solutions of this problem of constructing resolvable quintics. The rest of the paper (33–41) contains the proof of, and some comments upon, the fact already intimated, that if my resolvent sextic be found to have a rational root, and if the sextic be reduced to a quintic by a division depriving it of the rational root in question, the resolvent of the new quintic will itself have the same rational root.

2. The general quintic is

$$ax^5 + 5bx^4 + 10cx^3 + 10dx^2 + 5ex + f = 0, \tag{1}$$

which, if $x = y - ba^{-1}$, takes the shorter form

$$y^5 + 10\gamma y^3 + 10\delta y^2 + 5\varepsilon y + \zeta = 0, \tag{2}$$

where
$$\left.\begin{aligned}\gamma &= a^{-2}(ac-b^2),\\ \delta &= a^{-3}(a^2d-3abc+2b^3),\\ \varepsilon &= a^{-4}(a^3e-4a^2bd+6ab^2c-3b^4),\\ \zeta &= a^{-5}(a^4f-5a^3be+10a^2b^2d-10ab^3c+4b^5).\end{aligned}\right\} \qquad (3)$$

We shall assume the coefficients to be rational, recollecting however that what we find true for rational coefficients must also be true for irrational coefficients, provided that the word "rational," wherever used, be so extended in meaning as to comprise the coefficients as well as all rational numbers, embracing the whole in one hypothetical domain of rationality. It is known that the general quintic is not solvable algebraically, that is to say, by the simple algebraic processes of addition, subtraction, multiplication, division, the raising of powers, and the extraction of roots. Some quintics are so solvable. Those which can be broken up into factors directly are called reducible. Those which can be broken up indirectly into five linear factors, through the determination of the several roots by the aid of a sextic resolvent, may be classed as resolvable. There are thus four classes of quintics: the resolvable-reducible class, which includes, with some others, those having five rational roots; the unresolvable-reducible class; the resolvable-irreducible class; and the unresolvable-irreducible class, the last not being solvable by radicals. We shall here have chiefly to do with resolvable equations, but a few preliminary words concerning the criteria of reducibility will not be out of place.

3. A quintic in y is reducible when it is divisible either by $y+m$ or by y^2+py+q. We may proceed first to test a quintic by trying to find a linear factor, failing which we may look for a quadratic factor. In this it will be assumed that the coefficient of y^5 is unity. (If it has any other positive integral value a, the differences hereafter spoken of will all be multiplied by a or by some factor of a.) If we have the equation, for example, $y^5-5y-3=0$, the usual test for a linear factor is to substitute successively for y its possible rational values, namely, 1, -1, 3, -3. It is better, however, to pursue another known method, by which at first only two values are substituted, 1 and -1. The given equation being $\phi(y)=0$, with coefficients made integral, we set down in order the numerical values of $\phi(-1)$, $\phi(0)$, $\phi(1)$, and inspect them to see whether they respectively possess integral divisors ascending in arithmetical progression, the common difference being 1. In the example

$y^5 - 5y - 3 = 0$, the three values are $\phi(-1) = 1$, $\phi(0) = -3$, $\phi(1) = -7$. These obviously do not possess respective divisors exhibiting the required progression, so that the equation has no linear factor. If, as another example, we take $\phi(y) = y^5 - 13y + 6 = 0$, we find $\phi(-1) = 18$, $\phi(0) = 6$, $\phi(1) = -6$, in which we may discern the divisors 1, 2, 3, causing us to suspect that $\phi(-2) = 0$, which is the case. If the first coefficient of $\phi(y)$ is a instead of 1, the common difference must be some factor of a. In searching for a quadratic factor I usually employ the following method, which may be novel,* and which applies to equations of other degrees as well as to the quintic.

4. In the example $y^5 - 5y - 3 = 0$, we had $\phi(-1) = 1$, $\phi(0) = -3$, $\phi(1) = -7$. Continuing, we have $\phi(2) = 19$, $\phi(3) = 225$, $\phi_4 = 1001$, $\phi_5 = 3097$, etc. Avoiding negative products, in order if possible to consider only positive divisors, we take the series as follows:

$$\begin{aligned}
\phi(2) &= 19 &&= 1 \times 19, \\
\phi(3) &= 225 &&= 5 \times 45 = 9 \times 25, \\
\phi(4) &= 1001 &&= 7 \times 143 = 11 \times 91 = 13 \times 77, \\
\phi(5) &= 3097 &&= 19 \times 163.
\end{aligned}$$

What we have now to do is to look among the respective divisors for an ascending progression: if the differences increase regularly by 2 (more generally, by twice some factor of a), there is a quadratic factor. Such a progression appears in 1, 5, 11, 19. Carrying it back two steps to $\phi(0)$, we have $\phi(1) = -1 \times 7$, $\phi(0) = -1 \times 3$. The factors of $\phi(y)$ are therefore $y^2 + py - 1$, $y^3 + ry^2 + sy + 3$. It will be observed that the complementary divisors, 3, 7, 19, 45, 91, etc., have as differences the series 4, 12, 26, 46, etc., which have as second differences 8, 14, 20, etc., the constant third difference being 6. In general, any equation of the n^{th} degree having integral coefficients, the first being 1, and having a quadratic factor, must show a series of divisors having 2 as their uniform second difference, the complementary divisors having $(n-2)!$ as their uniform difference, of degree $n - 2$. (If the first coefficient is $a = mp$ instead of 1, the uniform differences will be $2m$ and $(n-2)!\,p$ respectively.) We may apply this system, obviously, in seeking for cubic or even larger factors, any factor of degree k yielding a series of which the k^{th} difference is always $k!$, or for the general

*Newton's method of factoring involves eventually only a series having uniform first differences, and to that extent is simpler, but it requires much more preparation.

form $m \,.\, k!$, according to an elementary principle of the theory of finite differences.

5. Thus far we have found only the final term of the quadratic or other factor. The theory of finite differences gives us the remaining coefficients of this factor at once. We have in fact found $\psi(0)$, $\psi(1)$, $\psi(2)$, etc., factors of $\phi(0)$, $\phi(1)$, $\phi(2)$, etc., from which, pursuing the same example, we establish the following scheme of differences for the quadratic factor:

y.	$\psi(y)$.	$\Delta\psi(y)$.	$\Delta^2\psi(y)$.
0	-1	0	2
1	-1	2	2
2	1	4	
3	5		

and for the cubic factor:

y.	$\chi(y)$.	$\Delta\chi(y)$.	$\Delta^2\chi(y)$.	$\Delta^3\chi(y)$.
0	3	4	8	6
1	7	12	14	6
2	19	26	20	
3	45	46		
4	91			

A known formula in finite differences is

$$f(y) = f(0) + y\Delta f(0) + \tfrac{1}{2}y(y-1)\Delta^2 f(0) + \frac{1}{2\,.\,3}y(y-1)(y-2)\Delta^3 f(0) + \ldots\,,$$

from which we find at once the quadratic factor of $x^5 - 5x - 3$ to be $-1 + y(y-1)$ and the cubic factor to be $3 + 4y + 4y(y-1) + y(y-1)(y-2)$. The same process may be followed, as suggested in the previous paragraph, in ascertaining factors of still higher degrees, when the given equation is, say, of the n^{th} degree, and the factors are respectively of the k^{th} and $(n-k)^{\text{th}}$.

6. If a quintic is reducible we do not need to inquire, except perhaps when we are looking for a case for purposes of illustration, whether it is also resolvable. If we find it to be irreducible, the construction of a resolvent and its examination for a rational root will determine the question of resolvability. It is known, however (see paragraph 16 farther on), that resolvable quintics have

either five real roots or one real and four imaginary. A quintic known to have just two imaginary roots may, whether reducible or not, be classed at once as unresolvable. Means for recognizing equations having two imaginary roots, often by mere inspection, may be found in a paper read by me before this Society at its summer meeting in 1894, and published in vol. XVII of the American Journal of Mathematics, its title being "A Method for Calculating Simultaneously all the Roots of an Equation." A glance, for example, at such quintics as $x^5+x^3-11x^2-2x+3=0$, $x^5-13x^3-9x+1=0$, $x^5-17x-1=0$, is enough to make sure that they are not resolvable. The following schedule will assist the reader in such cases, it being understood that there must be a marked distinction between the large or "dominant" coefficients and the rest, which must be relatively unimportant; and if such a distinction does not already exist, the equation must be so transformed linearly as to create it. In this schedule the unimportant coefficients are indicated by dots, the coefficient of x^5 by 1, and the other dominant coefficients by D or, when the sign is important, by $+$ or $-$ or $\pm$ or $\mp$.

ARRANGEMENTS OF COEFFICIENTS WHICH INDICATE TWO IMAGINARY ROOTS.

x^5	x^4	x^3	x^2	x	1
1	.	$+$	D	D	D
1	.	$+$	$\pm$	.	$\mp$
1	.	$+$	.	$-$	D
1	.	.	$\pm$	.	$\mp$
1	.	.	D	D	D
1	.	.	.	$-$	D
1	$\pm$	.	$\pm$	.	$\mp$
1	$\pm$	.	$\pm$	D	D
1	D	.	.	D	D
1	$\pm$	.	.	.	$\mp$
1	.	$-$	.	$-$	D
1	D	$\pm$	.	$\pm$	D
1	D	D	.	.	D
1	.	$-$	.	.	D
1	D	D	$\pm$	.	$\pm$
1	.	$-$	$\pm$	.	$\pm$
1	$\pm$	.	$\mp$	.	$\mp$

The word "span" is used, in the paper cited, for the space, or difference in the degrees of the exponents, between one dominant and the next. Thus, in the first form given in the foregoing schedule, there is a quadratic span followed by three linear spans. An "unlike span" is bounded by two dominants of unlike signs, and a "like span" by two dominants of like signs. The key to the schedule consists in the observation that two imaginary roots are, in a quintic, indicated always either by a cubic span, a like quadratic, or an unlike quartic.

7. Having noted certain methods for detecting equations which are either reducible or known to be unresolvable because having just two imaginary roots, we shall hereafter confine our attention to resolvable equations. To resolve any resolvable quintic of the form (2) it is sufficient, and when the quintic is irreducible it is necessary, to assume $y = u_1 + u_2 + u_3 + u_4$ and to determine the values of the four u's.* The paper of 1885 already referred to, "Analysis of Quintic Equations," contains not only a new resolvent, but also immediate formulæ expressing the roots of the quintic when a root of the resolvent is known. While these formulæ were derived in a manner not devoid of utility,† the method of proof now to be presented will be found far simpler. The expression for r_2 is also simplified greatly.

8. Let there be four quantities, called elements, namely, u_1, u_2, u_3, u_4, and let certain functions of these elements be defined as follows:

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*Those critical cases in which one or more of the u's disappear are treated in the earlier paper, paragraphs 7 and 33, and will receive some attention farther on.

†I may be excused for citing the judgment on this point of one or two competent critics. Says Cayley (Collected Mathematical Papers, IV, 612): "McClintock completes in a very elegant manner the determination of the roots of the quintic equation .... I reproduce this solution." Cayley's reproduction, or rather paraphrase, occupies nearly five of his quarto pages. The key to every improvement which I made in 1884 and 1885 lay in the recognition of the rational character of the quantities v and s, which for earlier writers were merely squares of quantities which they employed, without regard to their irrationality, as fundamental features of their systems, and to which I attributed distinct symbols; in the discovery and use of the all-important rational quantity t, connected with v and s by the relation $s = t^2 v$, and in the employment (which I recommended urgently by various arguments) of the rational symbols t and v in the discussion of the mechanism of the quintic and in the formulation of the two fundamental equations. It gave me much satisfaction to find these improvements shortly afterwards adopted by Professor Young (American Journal, X, 114), whose y and $-\frac{1}{2}t$ corresponded to my v and t. His formulæ for the roots, p. 114, corresponded with my Nos. 84 and 82, vol. VIII, pp. 67–8.

$$y = u_1 + u_2 + u_3 + u_4, \tag{4}$$
$$\gamma = -\tfrac{1}{2}(u_1u_4 + u_2u_3), \tag{5}$$
$$v^{\frac{1}{2}} = \tfrac{1}{2}(u_1u_4 - u_2u_3), \tag{6}$$
$$\delta = -\tfrac{1}{2}(u_2^2u_1 + u_3^2u_4 + u_1^2u_3 + u_4^2u_2), \tag{7}$$
$$t = \tfrac{1}{2}v^{-\frac{1}{2}}(u_2^2u_1 + u_3^2u_4 - u_1^2u_3 - u_4^2u_2), \tag{8}$$
$$p = v^{\frac{1}{2}}(u_2^2u_1 - u_3^2u_4)(u_1^2u_3 - u_4^2u_2), \tag{9}$$
$$\varepsilon = \gamma^2 + 3v - u_1^3u_2 - u_4^3u_3 - u_2^3u_4 - u_3^3u_1, \tag{10}$$
$$r_1 = u_1^5 + u_2^5 + u_3^5 + u_4^5, \tag{11}$$
$$r_2 = u_1^5 + u_4^5 - u_2^5 - u_3^5, \tag{12}$$
$$q_1 = \tfrac{1}{2}(u_1^5 - u_4^5), \tag{13}$$
$$q_2 = \tfrac{1}{2}(u_2^5 - u_3^5), \tag{14}$$
$$s_1 = \tfrac{1}{2}(q_1^2 + q_2^2), \tag{15}$$
$$s_2 = \tfrac{1}{2}(q_1^2 - q_2^2), \tag{16}$$
$$\zeta = -r_1 - 20tv. \tag{17}$$

The following relations may be proved at once by mere expansion in terms of the elements:

$$y^5 + 10\gamma y^3 + 10\delta y^2 + 5\varepsilon y + \zeta = 0, \tag{18}$$
$$p = \gamma\delta^2 - \gamma t^2v + (v - \gamma^2)(\varepsilon - \gamma^2 - 3v), \tag{19}$$
$$p^2 = (\delta^2 - t^2v)^2v + 8(\gamma^2 - v)(\gamma\delta^2 + \gamma t^2v + 2\delta tv)v + 16(\gamma^2 - v)^3v, \tag{20}$$
$$25tv^2 + (\zeta - \varepsilon t + \delta t^2 - \gamma t^3 - 10\gamma^2t)v - \gamma t(\gamma\varepsilon - \gamma^3 - \delta^2) - \gamma^2\zeta + 2\gamma\delta\varepsilon - \delta^3 = 0, \tag{21}$$
$$r_2 = [(\zeta + t\varepsilon)\gamma - (\varepsilon + t\delta)\delta + (\delta + t\gamma)(v - \gamma^2) + 12\gamma tv + t^3v]\,v^{-\frac{1}{2}}, \tag{22}$$
$$s_1 = \tfrac{1}{16}(r_1^2 + r_2^2) + \gamma^5 + 10\gamma^3v + 5\gamma v^2, \tag{23}$$
$$s_2 = \tfrac{1}{8}r_1r_2 - (5\gamma^4 + 10\gamma^2v + v^2)v^{\frac{1}{2}}, \tag{24}$$
$$\left.\begin{aligned} u_1^5 &= \tfrac{1}{4}r_1 + \tfrac{1}{4}r_2 + \sqrt{(s_1 + s_2)}, \\ u_2^5 &= \tfrac{1}{4}r_1 - \tfrac{1}{4}r_2 + \sqrt{(s_1 - s_2)}, \\ u_3^5 &= \tfrac{1}{4}r_1 - \tfrac{1}{4}r_2 - \sqrt{(s_1 - s_2)}, \\ u_4^5 &= \tfrac{1}{4}r_1 + \tfrac{1}{4}r_2 - \sqrt{(s_1 + s_2)}. \end{aligned}\right\} \tag{25}$$

Strictly speaking, only the first, numbered (18), of these relations requires proof by substitution of the elements of which the symbols are functions. No. (22) may be derived from the value of r_2 stated (paragraphs 3 and 31) in the paper of 1885, viz.

$$\left.\begin{aligned} r_2 = (\gamma^2 - v)^{-1}(&12\gamma tv^2 - \delta v^2 - t^3v^2 + 4\gamma^3tv + 2\gamma^2\delta v \\ &+ \gamma\delta t^2v + \delta^2tv - 2\gamma\varepsilon tv + \delta\varepsilon v - \gamma^4\delta + \gamma^2\delta\varepsilon - \gamma\delta^3)\,v^{-\frac{1}{2}}, \end{aligned}\right\} \tag{26}$$

by subtracting from the second member of (26) multiplied by $(\gamma^2 - v)$ the first member of (21) multiplied by $\gamma v^{-\frac{1}{5}}$, and dividing the remainder by $(\gamma^2 - v)$; and (26) was shown to be derived from some of the equations here numbered (5–19). The other relations may likewise be derived, as before, from those equations (5–17) here introduced as definitions, but I make special mention of (22), because it presents a notable simplification of the longest one of the formulæ given in the earlier paper for the final exhibition of the roots of the quintic. By eliminating p from (19) and (20) we have, as before,

$$\left.\begin{aligned} 25v^3 &+ (-t^4 + 14\gamma t^2 + 16\delta t - 35\gamma^2 - 6\varepsilon)\, v^2 \\ &+ (-2c_0 t^2 + 2\gamma\delta^2 + 4\gamma^2\varepsilon + 11\gamma^4 + \varepsilon^2)\, v - c_0^2 = 0, \end{aligned}\right\} \tag{27}$$

where $c_0 = -\gamma^3 + \gamma\varepsilon - \delta^2$.

9. In (18) we have (2), the shortened form of the general quintic, for the solution of which it is therefore necessary to determine the elements from the element-formulæ here numbered (25), by the extraction of fifth-roots, and for the employment of those formulæ we require to know the values of t and v, a subject to be considered further on. One value of each element is obtained without the intervention of the fifth-roots of unity, and such values will be real if the right-hand members of (25) are real; let the values so obtained be used in (4) for the determination of one root of (18), which let us designate as y_1. Before discussing the other roots of the quintic it is desirable to note the relations which exist between the elements.

10. That it is not necessary to determine more than one of the elements by the extraction of a fifth-root has long been known. It has also been shown by researches in the theory of substitutions that the root of the resolvable quintic may have the form $y_1 = u_1 + z_2 u_1^2 + z_3 u_1^3 + z_4 u_1^4$, where only u_1 involves a fifth-root. According to Schläfli,* $u_2 = u_1^2 . u_4^2 u_2 (u_1 u_4)^{-2}$, $u_3 = u_1^2 u_3 . u_1^{-2}$, $u_4 = u_1 u_4 . u_1^{-1}$. These expressions are almost exactly what we want, so that, modifying two of them slightly, we have now

$$\left.\begin{aligned} u_2 &= z_2 u_1^2 = u_4^2 u_2 (u_1 u_4)^{-2} . u_1^2, \\ u_3 &= z_3 u_1^3 = u_1^2 u_3 . u_1^{-5} . u_1^3, \\ u_4 &= z_4 u_1^4 = (u_1 u_4)\, u_1^{-5} . u_1^4. \end{aligned}\right\} \tag{28}$$

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*Cited 1885 in paragraph 19.

The value of u_1^5 is known, by (25), and those of u_1u_4, $u_4^2u_2$, and $u_1^2u_3$ are obtainable from known quantities by the aid of (5–8), it being observed that $u_1u_2u_3u_4 = \gamma^2 - v$ and that $(u_1^2u_3 - u_4^2u_2)^2 = (u_1^2u_3 + u_4^2u_2)^2 - 4u_1u_2u_3u_4 \,.\, u_1u_4$; that is to say,

$$\left.\begin{aligned} u_1^2u_3 &= -\tfrac{1}{2}(\delta + tv^{\frac{1}{2}}) + \tfrac{1}{2}\sqrt{[(\delta + tv^{\frac{1}{2}})^2 - 4(\gamma^2 - v)(v^{\frac{1}{2}} - \gamma)]}, \\ u_4^2u_2 &= -\tfrac{1}{2}(\delta + tv^{\frac{1}{2}}) - \tfrac{1}{2}\sqrt{[(\delta + tv^{\frac{1}{2}})^2 - 4(\gamma^2 - v)(v^{\frac{1}{2}} - \gamma)]}. \end{aligned}\right\} \tag{29}$$

Let us now turn to the question of ascertaining the other four roots of the quintic, having found $y_1 = u_1 + u_2 + u_3 + u_4 = u_1 + z_2u_1^2 + z_3u_1^3 + z_4u_1^4$.

11. Instead of taking u_1 as the fifth-root of u_1^5, let us take ωu_1, where ω is an imaginary fifth-root of unity, and let us denote the corresponding root of the quintic by y_5. Then $y_5 = \omega u_1 + z_2\omega^2u_1^2 + z_3\omega^3u_1^3 + z_4\omega^4u_1^4 = \omega u_1 + \omega^2u_2 + \omega^3u_3 + \omega^4u_4$. Proceeding in like manner, we derive the following schedule, the remaining fifth-roots of unity being ω^2, ω^3, ω^4, regard being had to the relations $\omega^5 = 1$, $\omega^6 = \omega$, etc.

$$\left.\begin{aligned} y_5 &= \omega u_1 + \omega^2u_2 + \omega^3u_3 + \omega^4u_4, \\ y_4 &= \omega^2u_1 + \omega^4u_2 + \omega u_3 + \omega^3u_4, \\ y_3 &= \omega^3u_1 + \omega u_2 + \omega^4u_3 + \omega^2u_4, \\ y_2 &= \omega^4u_1 + \omega^3u_2 + \omega^2u_3 + \omega u_4, \\ y_1 &= u_1 + u_2 + u_3 + u_4. \end{aligned}\right\} \tag{30}$$

If we multiply the fourth line by ω, the third by ω^2, and so on, and add all five together, recollecting that $1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$, we derive $5u_1 = y_1 + \omega y_2 + \omega^2y_3 + \omega^3y_4 + \omega^4y_5$. If we proceed similarly with ω^2, ω^4, etc.; then with ω^3, ω^6, etc.; then with ω^4, ω^8, etc., we have finally the well-known definitions of the elements according to the theory of Bezout and Euler, $5u_m = y_1 + \omega^my_2 + \omega^{2m}y_3 + \omega^{3m}y_4$, where m is 1, 2, 3, or 4. We have thus ended this sketch of a new theory of the quintic by exhibiting the elements as functions of the roots, having begun by defining y as the sum of four elements, in precisely the reverse of the usual order.*

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* This reverse method will be found applicable to equations of all degrees. Thus, for the cubic, we may begin by defining $y = u_1 + u_2$, $\gamma = -u_1u_2$, $\delta = -u_1^3 - u_2^3$, and then prove by substitution that $y^3 + 3\gamma y + \delta = 0$; then, as $(u_1^3 - u_2^3)^2 = \delta^2 + 4\gamma^3$, it follows that $u_1^3 = -\frac{1}{2}\delta + \frac{1}{2}\sqrt{(\delta^2 + 4\gamma^3)}$, and $u_2^3 = -\frac{1}{2}\delta - \frac{1}{2}\sqrt{(\delta^2 + 4\gamma^3)}$. Again, for the biquadratic, let $y = u_1 + u_2 + u_3$, $\gamma = -\frac{1}{3}(u_1^2 + u_2^2 + u_3^2)$, $\delta = -2u_1u_2u_3$, $\varepsilon = u_1^4 + u_2^4 + u_3^4 - 2u_1^2u_2^2 - 2u_1^2u_3^2 - 2u_2^2u_3^2$, whence $y^4 + 6\gamma y^2 + 4\delta y + \varepsilon = 0$. Then, since $u_1^2 + u_2^2 + u_3^2 = -3\gamma$, and $u_1^2u_2^2 + u_1^2u_3^2 + u_2^2u_3^2 = \frac{1}{4}(9\gamma^2 - \varepsilon)$, and $u_1^2u_2^2u_3^2 = \frac{1}{4}\delta^2$, we may assign a cubic equation of which the roots shall be u_1^2, u_2^2, and u_3^2, namely, $u^6 + 3\gamma u^4 + \frac{1}{4}(9\gamma^2 - \varepsilon)u^2 - \frac{1}{4}\delta^2 = 0$. While by this method certain relations are assumed in the definitions, it has the advantages of lucidity and succinctness in exhibiting the mechanism of solution.

12. By substituting in any of the definitions (5–17) the values of the elements, just found, in terms of the roots, we shall have the quantities in question exhibited as functions of the roots. In this way we may derive the customary expressions for $\gamma, \delta, \varepsilon, \zeta$, but we are now particularly concerned with v and t. Thus,

$$25\,(u_1u_4 - u_2u_3) = 50v^{\frac{1}{2}} = (\omega + \omega^4 - \omega^2 - \omega^3)(y_1y_2 + y_2y_3 + y_3y_4 + y_4y_5 + y_5y_1 - y_1y_3 - y_2y_4 - y_3y_5 - y_4y_1 - y_5y_2). \quad (31)$$

If ϕ represent the latter bracket, we have (since $\omega + \omega^4 - \omega^2 - \omega^3 = \pm\sqrt{5}$) $50v^{\frac{1}{2}} = \pm\sqrt{5}\,.\,\phi$, whence $500v = \phi^2$. Any other system of designating the subscripts of y in (30) will produce in (31) one or other of the six forms in which $\pm\phi$, and therefore v, can be expressed as a function of the roots. The denominator of t, as defined in (8), is $2v^{\frac{1}{2}} = \pm 5^{-2}\sqrt{5}\,.\,\phi$, and its numerator is $u_2^2u_1 + u_3^2u_4 - u_1^2u_3 - u_4^2u_2$, the value of which may similarly be found to be $\pm 5^{-2}\sqrt{5}\,.\,(y_1y_2y_5 + y_2y_3y_1 + y_3y_4y_2 + y_4y_5y_3 + y_5y_1y_4 - y_1y_3y_5 - y_2y_4y_1 - y_3y_5y_2 - y_4y_1y_3 - y_5y_2y_4)$ or say $\pm 5^{-2}\sqrt{5}\,.\,\sigma$, so that $t = \sigma\phi^{-1}$. The latter operation is however somewhat intricate and requires special consideration.

13. Let the letter c represent the sum of five similar functions of the roots, comprising a cycle, each function being formed from the one preceding by advancing the subscript of each root involved, y_1 becoming y_2, y_2 becoming y_3, ... y_5 becoming y_1. Thus $\phi = \mathrm{c}y_1y_2 - \mathrm{c}y_1y_3 = \mathrm{c}y_1(y_2 - y_3)$, and $\sigma = \mathrm{c}y_1y_2y_5 - \mathrm{c}y_1y_3y_5 = \mathrm{c}y_1(y_2 - y_3)\,y_5$. As already stated, the substitution in (6) of the values of the elements in terms of the roots produces $50v^{\frac{1}{2}} = \pm\sqrt{5}\,.\,\phi$; but a similar substitution in the numerator of t in (8) does not produce at once $\pm 5^{-2}\sqrt{5}\,.\,\sigma$, but $\pm 5^{-3}\sqrt{5}\,.\,(4\sigma + \xi)$, where $\xi = \mathrm{c}y_1^2(y_3 + y_4 - y_2 - y_5)$, and it is necessary to show that $\xi = \sigma$. Since $y_1 + y_2 + y_3 + y_4 + y_5 = 0$, we have $\xi = \mathrm{c}y_1(y_2 + y_5 + y_3 + y_4)(y_2 + y_5 - y_3 - y_4) = \mathrm{c}y_1(y_2^2 + y_5^2 - y_3^2 - y_4^2 + 2y_2y_5 - 2y_3y_4) = 2\sigma - \xi$, whence $\xi = \sigma$. For, in ξ, $\mathrm{c}y_1^2y_3 = \mathrm{c}y_1y_4^2$, $\mathrm{c}y_1^2y_4 = \mathrm{c}y_1y_3^2$, $\mathrm{c}y_1^2y_2 = \mathrm{c}y_1y_5^2$, and $\mathrm{c}y_1^2y_5 = \mathrm{c}y_1y_2^2$; and, in σ, $\mathrm{c}y_1y_3y_5 = \mathrm{c}y_1y_3y_4$.—Or, we may prove that $\xi - \sigma = 0$ by showing that $\xi - \sigma$ is exactly divisible by $y_1 + y_2 + y_3 + y_4 + y_5 = 0$.

14. For ascertaining the values of t and v, which is all that is necessary in order to exhibit the elements of the roots as in (25), I have nothing to add to

my earlier discoveries, contained in paragraphs 3, 4, 5, 25, 41, 42, of the paper of 1885, which will be summarized in this paragraph for the convenience of the reader. Let there be an auxiliary quintic, $A_y = t^5 + 10\gamma t^3 + 10\delta t^2 + 5\varepsilon t + \zeta$, with its canonizant, $C_y = c_0 t^3 + c_1 t^2 + c_2 t + c_3$, and its simplest linear covariant, $L_y = l_0 t + l_1$, where

$$\left.\begin{aligned}
c_0 &= -\gamma^3 + \gamma\varepsilon - \delta^2, \\
c_1 &= -\gamma^2\delta + \gamma\zeta - \delta\varepsilon, \\
c_2 &= -\gamma\delta^2 + \gamma^2\varepsilon + \delta\zeta - \varepsilon^2, \\
c_3 &= \quad 2\gamma\delta\varepsilon - \gamma^2\zeta - \delta^3,
\end{aligned}\right\} \tag{32}$$

$$\left.\begin{aligned}
l_0 &= -15\gamma^4\varepsilon + 10\gamma^3\delta^2 - 2\gamma^2\delta\zeta + 14\gamma^2\varepsilon^2 - 22\gamma\delta^2\varepsilon + \gamma\zeta^2 + 9\delta^4 - 2\delta\varepsilon\zeta + \varepsilon^3, \\
l_1 &= \quad 9\gamma^4\zeta - 20\gamma^3\delta\varepsilon + 10\gamma^2\delta^3 + 8\gamma^2\varepsilon\zeta - 12\gamma\delta\varepsilon^2 - 2\gamma\delta^2\zeta + 6\delta^3\varepsilon + \delta\zeta^2 - \varepsilon^2\zeta.
\end{aligned}\right\} \tag{33}$$

Let a value of t be found by the numerical solution of my resolvent,

$$A_y L_y - 25C_y^2 = 0, \tag{34}$$

which may be written thus:

$$\left.\begin{aligned}
&(l_0 - 25c_0^2)\,t^6 + (l_1 - 50c_0c_1)\,t^5 + 5\,(2\gamma l_0 - 5c_1^2 - 10c_0c_2)\,t^4 \\
&+ 10\,(\gamma l_1 + \delta l_0 - 5c_0c_3 - 5c_1c_2)\,t^3 + 5\,(2\delta l_1 + \varepsilon l_0 - 5c_2^2 - 10c_1c_3)\,t^2 \\
&+ (5\varepsilon l_1 + \zeta l_0 - 50c_2c_3)\,t + \zeta l_1 - 25c_3^2 = 0.
\end{aligned}\right\} \tag{35}$$

Also, after t is known, let v be found by using either of these expressions:

$$v = -C_y A_y^{-1} = -\tfrac{1}{25} L_y C_y^{-1}. \tag{36}$$

For the broader form (1) of the general quintic, namely, $ax^5 + 5bx^4 + \ldots\ldots = 0$, where $x = y - ba^{-1}$, let there be an auxiliary quintic in $\tau = t - ba^{-1}$, namely $A = a\tau^5 + 5b\tau^4 + \ldots\ldots$, with its canonizant C and its simplest linear covariant L, and we may obtain τ by means of my broader resolvent, of which (34) is a special case,

$$AL - 25C^2 = 0, \tag{37}$$

and v from

$$v = -CA^{-1}a^{-2} = -\tfrac{1}{25} LC^{-1}a^{-2}. \tag{38}$$

It will be found that ϕ is the same function of x as of y, but this is not true of σ, which is equivalent to $t\phi$. Let $\tau\phi = \psi$; then $\psi = t\phi - ba^{-1}\phi = \sigma - ba^{-1}\phi$, and $\tau = \psi\phi^{-1}$, where ψ is the same function of the roots of (1) as σ is of the roots of (2). As τ is a function of the roots which is the reciprocal of the same function of the reciprocals of the roots, it is a covariant function, and having six values is to be determined by a covariant sextic (37), equivalent to $(\phi_1\tau - \psi_1)(\phi_2\tau - \psi_2)\ldots\ldots(\phi_6\tau - \psi_6) = 0$.

15. It is well known that to resolve a resolvable irreducible quintic it is necessary and sufficient to assume that y is the sum of four elements and to determine the values of the elements. It was proved by Serret that no irreducible quintic is resolvable unless Lagrange's resolvent sextic in r_1 has a rational root. Now r_1 is only rational when tv is rational, by (17); and since (21) and (27) may be expressed in terms of t and tv, whence by elimination t may be expressed rationally in terms of tv, and *vice versa*, it follows that (certain critical cases, as for example $t = \infty$, excepted) no such quintic is resolvable unless (34), and therefore unless (37), has a rational root. Again, from (36) and (38), no such quintic is resolvable unless v has a rational value, and therefore unless Malfatti's resolvent in v* has a rational root. Again, no resolvent is valid unless it has a rational root in every resolvable case, and no quantity can serve as a resolvent unless that quantity and t are respectively rational functions of each other. Since t and v cannot be simplified further as functions of the elements and of the roots, it follows that no other simple resolvent of the quintic can ever be discovered.† All this comes at once from the theory of Galois.

16. It was pointed out in the earlier paper, paragraph 53, that if t (or v) has two equal rational values, v (or t) is not necessarily rational, and the reason was given. Also, "by referring to the definition of v as a function of the roots,

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* The other resolvent, that of Malfatti, the only one known prior to the discovery of (34) and (37), was first presented as a resolvent in v by me, Malfatti's own unknown quantity being equivalent to $25v + 5\gamma^2 + \frac{5}{3}\varepsilon$. The example so set has been followed unconsciously by Giudice (Atti, Accad. di Torino, 1892, XXVII, 817–826), whose ω is our v. Cayley (IV, 610, etc.) avoided recognition of the quantity v, but ascribed to me a resolvent in $w = 500v$. Giudice cites this volume of Cayley, but having himself rediscovered v, makes no mention of the resolvent in "w."

† The well-known resolvent in ϕ, suggested by Jacobi and elaborated by Cayley, is a form of Malfatti's resolvent which is very simple when ϕ happens to have a rational value, but is useless in all other resolvable cases. Cayley suggested also a generalized, but still more useless resolvent in $\Phi = aU^{-1}(x\phi - y\psi)$, where U is the quintic and x and y its variables. This has no wider validity than the sextic in ϕ, while far more complex. In republishing his paper (IV, 323) he availed himself of the strikingly simple form (37) into which I had meanwhile succeeded in throwing the continued product of the six values of ϕ, a product employed by him in the final term of his sextic in Φ, and in his note (p. 610) explained that the change "was in fact suggested by McClintock's formula." Although his comments on other points were warmly favorable—he cites (38), for example, as "a very important and remarkable theorem"—he seems not to have approved my arguments (see especially paragraph 17 of the paper of 1885) for the use of the always rational v; and throughout his long reproduction of my discussion he avoided and ignored this quantity v.

it will be found that all of its values are real when all five roots are real, that some of its values are real when only one root is real, and that when three roots are real and unequal all of the six values of v are unreal, unless two are equal as just explained." That is to say, quintics having just two imaginary roots are not resolvable when the real roots are unequal.

17. Thus far we have been going chiefly over old ground, with some improvement, I trust, in the mode of stating certain important parts of the theory. I have now to bring forward matters which will be found entirely novel, and of high importance in the theory of the quintic. It has long been known that some quintics are resolvable, and the former paper supplied immediate means for effecting the solution of all such resolvable equations. The problem which I have since attacked is that of the algebraic definition of the form of the general resolvable quintic, which includes the power of constructing at will the coefficients of resolvable quintic equations. The solution of this problem is now presented. Assign rational values to four parameters, q, r, t, w, and let

$$l = [t^2w^2 - r^2(1+w^2)] \div 4(q+1)[q^2(1+w^2)-1]. \tag{39}$$

Then

$$\left.\begin{aligned} v &= l^2 \div (1+w^2), \\ \gamma &= ql, \\ \delta &= (r+t)\,l, \\ \varepsilon &= \gamma^2 + v(3+4w) + (\gamma^2 - v)^{-1}(\gamma\delta^2 - \gamma t^2 v - 2\delta t v w), \end{aligned}\right\} \tag{40}$$

and the form of ζ is determined by (21), viz.

$$\zeta = (\gamma^2 - v)^{-1}(25tv^2 - \gamma t^3 v + \delta t^2 v - \varepsilon t v - 10\gamma^2 t v + \gamma^4 t - \gamma^2 \varepsilon t + \gamma\delta^2 t + 2\gamma\delta\varepsilon - \delta^3). \tag{41}$$

For example, let us assume $q = -\frac{1}{2}$, $r = -1$, $t = 1$, $w = -1$, so that $l = 1$, $v = \frac{1}{2}$, $\gamma = -\frac{1}{2}$, $\delta = 0$, $\varepsilon = -\frac{5}{4}$, $\zeta = -25$; the equation (2) is $y^5 - 5y^3 - \frac{25}{4}y - 25 = 0$, and the resolvent (35), after dividing throughout by the factor $\frac{25}{16}$, is

$$-206t^6 - 244t^5 - 1470t^4 + 320t^3 - 1325t^2 + 4950t - 2025 = 0,$$

which has the rational root $t = 1$. For ascertaining, conversely, the corresponding value of v, by the aid of (36), assuming the constitution of the equation in y to be unknown, we have $A_y = -\frac{141}{4}$, $C_y = \frac{141}{8}$, so that $v = -C_y A_y^{-1} = \frac{1}{2}$. Then, referring to (40), we find $w = -1$ by the aid of the expression for ε, then

$l^2 = 1$, whence if $l = 1$ we get $q = -\frac{1}{2}$ and $r = -1$; and that $l = 1$, and not -1, is corroborated by (39). In this way the original parameters are rediscovered. Or, more directly, we may find the parameters of resolvable quintics in general, including those of the quintic just taken as an illustration, by putting

$$l = \frac{\delta^2 + t^2 v + 4\gamma(\gamma^2 - v)}{2\delta t - 4(\gamma^2 - v)}, \tag{42}$$

deriving now the values of q, r, w, from (40). That the two expressions for l, (39) and (42), are consistent will be seen upon substituting for the symbols γ, δ, v, in (42) their values in terms of q, r, s, t, l, as given in (40). If we define l as in (42), $q = \gamma l^{-1}$, $r = \delta l^{-1} - t$, and w as determined by (40, ε), it remains only to prove that $v = l^2(1 + w^2)^{-1}$ in order to establish all the statements of this paragraph.

18. That $l^2 = v(1 + w^2)$ may be proved either algebraically, employing only the rational symbols γ, δ, t, p, v; or by defining w and l in terms of the elements. Let w be such that $p = wv[2\delta t - 4(\gamma^2 - v)]$. This substituted in (19) produces (40, ε). This expression for p, again, substituted in (20), gives $w^2v^2[2\delta t - 4(\gamma^2 - v)]^2 =$ second member; and the addition to each side of $v^2[2\delta t - 4(\gamma^2 - v)]^2$ transforms the second member into a perfect square multiplied by v, that is to say,

$$v^2(1 + w^2)[2\delta t - 4(\gamma^2 - v)]^2 = v[\delta^2 + t^2 v + 4\gamma(\gamma^2 - v)]^2. \tag{43}$$

Hence, if l be defined as in (42), $v(1 + w^2) = l^2$. Let us now consider the other mode of proof, by which w and l are defined as functions of the elements.

19. Let $m = [u_1^2 u_3 - u_4^2 u_2] \div [u_2^2 u_1 - u_3^2 u_4]$, and let

$$\left.\begin{aligned} w &= \frac{2m}{m^2 - 1}, \\ l &= v^{\frac{1}{2}} \frac{m^2 + 1}{m^2 - 1}. \end{aligned}\right\} \tag{44}$$

Then $1 + w^2 = \left(\frac{m^2 + 1}{m^2 - 1}\right)^2$, and $l^2 = v(1 + w^2)$. That

$$w = \frac{\gamma(\delta^2 - t^2 v) + (\gamma^2 - v)(\gamma^2 + 3v - \varepsilon)}{v[2\delta t - 4(\gamma^2 - v)]}, \tag{45}$$

and

$$l = \frac{\delta^2 + t^2 v + 4\gamma(\gamma^2 - v)}{2\delta t - 4(\gamma^2 - v)}, \tag{42}$$

may be verified by substituting for all the symbols involved their values in terms of the elements. Therefore w and l are rational when t is rational, though m is usually irrational. Then q and r must be defined, as in (40), in terms of γ, δ, t, l, that is to say, $q=\gamma l^{-1}$, $r=\delta l^{-1}-t$. A simple illustration is supplied by $u_1=1$, $u_2=-1$, $u_3=\frac{1}{2}$, $u_4=2$, $\gamma=-\frac{3}{4}$, $\delta=1$, $\varepsilon=\frac{33}{8}$, $m=9$, $w=\frac{9}{40}$, $l=\frac{41}{32}$, $v=\frac{25}{16}$, $t=2$. Since $5u=y_1+\omega^n y_2+\omega^{2n}y_3+\omega^{3n}y_4$, where n is 1, 2, 3, or 4, and ω is a fifth-root of unity, as in paragraph 11, it becomes a mere matter of substitution to express w and l, and therefore q and r, as rational functions of the roots of the quintic. The rational parameter t is a rational function of the roots and also of the elements, and is obtained from the coefficients by means of the resolvent; the other three rational parameters are rational functions of the roots, and also of the elements, and also of the coefficients and t; while the coefficients are, as here assigned, rational functions of the four rational parameters, as well as of the usually irrational roots and of the usually irrational elements. For the fuller form (1) of the quintic a fifth rational parameter is obviously to be assigned, namely, the quantity ba^{-1}.

20. Having now at our disposal the skeleton, so to speak, of the general resolvable quintic, we can adjust the parameters so as to produce at will resolvable quintics having specific forms or properties. When $q=0$, $\gamma=0$; when $r=-t$, $\delta=0$; when m is rational, $1+w^2=\left(\frac{m^2+1}{m^2-1}\right)^2$ because $w=\frac{2m}{m^2-1}$, so that v is the square of a rational quantity, and so on. If, for example, we give to w a rational value such that $w^2=5n^2-1$, where n is rational, we shall have $\frac{1}{5}l^2n^{-2}=v=\frac{1}{500}\phi^2$, whence $\phi=10ln^{-1}$; and in this case ϕ has a rational value, and the historic resolvent of Jacobi and Cayley becomes available, as having a rational root. In order, therefore, to construct a quintic resolvable by ϕ, we must take rational values for w and n such that $w^2=5n^2-1$, the other three parameters, q, r, t, remaining at our arbitrary disposal. How small a proportion of all resolvable quintics are resolvable by ϕ may be appreciated when we reflect that, taking for the moment integral values only, every such value given to w produces a resolvable quintic, while of the numbers up to 100 only 2 and 38 appear to be admissible for a value of w consistent with the resolvent in ϕ.—In certain critical cases, which we shall now examine, the system of construction presented in (39) and (40) requires modification.

21. Critical Case A: $\phi = 0$, $v = 0$, $t = \sigma\phi^{-1} = \infty$, $q = q_a t$, $r = r_a t$, $l_a = lt$. This case occurs when $u_1 u_4 = u_2 u_3$. To construct a resolvable quintic having these properties we have disposable three rational parameters, q_a, r_a, w, and assigning values to these we obtain the value of l_a (see 39) from

$$\begin{aligned} l_a = lt &= [t^3 w^2 - r_a^2 t^3 (1 + w^2)] \div 4 q_a t q_a^2 t^2 (1 + w^2) \\ &= [w^2 - r_a^2 (1 + w^2)] \div 4 q_a^3 (1 + w^2); \end{aligned} \tag{46}$$

then, from (40), since $l = l_a t^{-1} = 0$, we have $v = 0$, $\gamma = q_a l_a$, $\delta = (r_a + 1) l_a$, $\varepsilon = \gamma^2 + \delta^2 \gamma^{-1} - s\gamma^{-1}$, where $s = t^2 v = l_a^2 \div (1 + w^2)$. For determining ζ we shall need to vary (41) by multiplying it throughout by $(\gamma^2 - v)^2$ and by substituting in it for ε the equivalent of ε as given by (40), whence we derive this modified expression for ζ, available for all values of t and v:

$$\begin{aligned} (\zeta + 22tv)(\gamma^2 - v)^2 &= (\gamma^2\delta - 2\gamma tv + \delta v)(\delta^2 - t^2 v) \\ &+ (\gamma^2 - v)(2\gamma^3\delta + 6\gamma\delta v + 8\gamma^2 tv) + (\gamma^2 t + tv - 2\gamma\delta)(2\delta t - 4\gamma^2 + 4v)\, vw. \end{aligned} \tag{47}$$

In the case now under consideration, $v = 0$, $tv = 0$, $t^2 v = s$, and (47) becomes, after dividing throughout by γ^4,

$$\zeta = 2\gamma\delta + \gamma^{-2}\delta(\delta^2 - s + 2sw). \tag{48}$$

If, for example, we assign $q_a = -\frac{2}{13}$, $r_a = \frac{3}{13}$, $w = \frac{5}{12}$, we have, by (46), $l_a = -\frac{13}{2}$, so that $\gamma = 1$, $\delta = -8$, $s = 36$, $\varepsilon = 29$, and, by (48), $\zeta = -480$, and we have constructed, for the special case $v = t^{-1} = 0$, the solvable equation $y^5 + 10y^3 - 80y^2 + 145y - 480 = 0$. This equation served, in the 1885 paper, as an illustration of a method then presented for solving equations for which $v = 0$, $t = \infty$, that is to say, equations for which the coefficient of t^6 in the resolvent is found to vanish, while the other coefficients do not. Conversely, when an equation of this sort has presented itself, and has been solved as to the value of s, we can determine the three parameters by the aid of (42), which in this case becomes

$$l_a = lt = (\delta^2 + s + 4\gamma^3) \div 2\delta, \tag{49}$$

which assists in finding q_a and r_a, and of (48), which supplies the value of w.*

22. Critical Case B: $\phi = 0$, $\sigma = 0$, t finite but indeterminate $= \sigma\phi^{-1}$, $v = 0$, $s = 0$. As t is in this case of no account whatever, we may allow it to

* The solution of this class of quintics, as published in the paper of 1885, consists in the formulæ $s = \gamma^3 - \gamma\varepsilon + \delta^2$, $r_1 = -\zeta$, $r_2 = s - \frac{1}{2}(4\gamma s - \gamma^4 - \delta\zeta + \varepsilon^2)$, $s_1 = \frac{1}{16}(r_1^2 + r_2^2) + \gamma^5$, $s_2 = \frac{1}{8} r_1 r_2$, these values being employed in (25) of the present paper.

vanish, knowing that similar results must follow, whatever value be assigned to t. In this case B we have, from (40) and (41),

$$\left.\begin{aligned}\varepsilon &= \gamma^2 + \gamma^{-1}\delta^2, \\ \zeta &= \gamma\delta + \gamma^{-1}\delta\varepsilon \cdot\end{aligned}\right\} \tag{50}$$

These novel expressions will enable us both to construct quintics of this class by assigning values to γ and δ, and to recognize such quintics, whenever they may present themselves, among those in which $v = 0$. In this remarkable case, not only does the first coefficient of the resolvent vanish, as in case A, but all the other coefficients likewise vanish identically. As regards the affiliation of these quintics to the general scheme of (39) and (40), I am disposed to divide them into two classes, those namely in which w is infinite, and those in which w is indeterminate. The latter class will be considered separately. As for the former, dropping t and putting $w^{-1} = 0$, we have from (39),

$$l = - r^2 \div 4\,(q^3 + q^2). \tag{51}$$

Since $\gamma = ql$ and $\delta = rl$, it may appear that this mode of construction is less general, as it certainly is less simple, than by merely assigning values to γ and δ; but this is not the fact, as will be seen on referring to (42), which becomes

$$l = -\gamma - \tfrac{1}{4}\delta^2\gamma^{-2}, \tag{52}$$

an expression which enables us to find rational values of q and r for any assigned values of γ and δ. Since $w^{-1} = 0$, we have, from (44), $m^2 - 1 = 0$, $m = \pm 1$; that is to say,

$$u_1^2u_3 - u_4^2u_2 = \pm\,(u_2^2u_1 - u_3^2u_4)\,. \tag{53}$$

Since $v = 0$ and $tv^{\frac{1}{5}} = 0$, we have, from (6) and (8),

$$\begin{aligned}&u_1u_4 = u_2u_3\,, \\ &u_2^2u_1 + u_3^2u_4 = u_1^2u_3 + u_4^2u_2\,.\end{aligned}$$

From these, if we take the upper sign in (53), we find that u_1, u_2, u_3, u_4, are in geometrical progression, say $u_2 = ku_1$, $u_3 = k^2u_1$, $u_4 = k^3u_1$; or, if we take the lower sign, we reach a similar progression, $u_1 = ku_3$, $u_4 = k^2u_3$, $u_2 = k^3u_3$. Whenever, therefore, the elements form a geometrical progression, we have a quintic of this sort, and *vice versa*. Let us, for example, take $k = -2^{\frac{1}{5}}$, $u_1 = 2^{\frac{1}{5}}$, $u_2 = -2^{\frac{2}{5}}$, $u_3 = 2^{\frac{3}{5}}$, $u_4 = -2^{\frac{4}{5}}$: from (6), (8), (44), we have $v = 0$, $tv^{\frac{1}{5}} = 0$, $\phi = 0$, $\sigma = 0$, $s = 0$, $w^{-1} = 0$; from (5), (7), we have $\gamma = 2$, $\delta = 2$; and from (50), $\varepsilon = 6$, $\zeta = 10$. If these values be substituted in the resolvent (34), every term

vanishes. Conversely, the values of k and u_1 can be derived readily from (5) and (7), which give $\gamma = -k^3u_1^2$, $\delta = -(1+k^5)k^2u_1^3$. The quintic of this class may thus be solved in a manner differing from the special solution given in paragraph 8 of the paper of 1885,* though not more simple.

23. CRITICAL CASE C: Same as case B, with the added restriction that $4\gamma^3 + \delta^2 = 0$, so that $\varepsilon = -3\gamma^2$, $\zeta = -2\gamma\delta$. In this case we have, as in the preceding paragraph, $4\gamma^3 = -4k^9u_1^6$, and $\delta^2 = (k^4 + 2k^9 + k^{14})u_1^6$, the sum being 0, whence $1 - k^5 = 0$, and k must be either 1 or ω, a fifth-root of unity. Correspondingly, u_1 must be either g or $g\omega$, where g is rational. In the one case, referring to (30), we have $y_1 = 4u_1$; in the other case, writing ωg for u_1 in (30), we have $y_1 = \omega g + \omega^2 g + \omega^3 g + \omega^4 g$, and $y_2 = 4g$. In each case there is one root $4g$, and four other equal roots, each $-g$. The form of these quintics is therefore $y^5 - 10g^2y^3 - 20g^3y^2 - 15g^4y - 4g^5 = 0$. This informs us that if the elements are to be identical they may have a rational value assigned to them, and that any quintic of this form must have identical elements. For them, as they are also comprised under case B, $v = 0$, $tv^{\frac{1}{5}} = 0$, and every term of the resolvent vanishes. My only reason for discussing them apart from case B is that they appear to require distinct consideration from the point of view of the constructive formulæ (39) and (40). If we refer to (44), we see that when $u_1 = u_2 = u_3 = u_4$, or when $u_2 = \omega u_1$, $u_3 = \omega^2 u_1$, $u_4 = \omega^3 u_1$, or in short when $k^5 = 1$, m becomes indeterminate, and thus w becomes indeterminate, instead of infinite as observed under case B. In case C we have thus both t and w indeterminate, while $v = 0$, and therefore $l = 0$. Referring to (39), we find that to bring this case under the general scheme we need to put $r = r_b q$, and $q = \infty$; then if $l_b = lq$, (39) becomes

$$l_b = -\tfrac{1}{4} r_b^2, \tag{54}$$

and we have $l = 0$, $v = 0$, $\gamma = l_b$, $\delta = r_b l_b$. Since $\gamma = -u_1^2$, it follows that $r_b = 2u_1$. It is to be inferred that in case B l has a value other than zero, and v, which has l in its numerator, vanishes because w, an infinite quantity, is contained in its denominator; and that in case C, as may be seen by referring to (52), l vanishes identically, while w becomes indeterminate. The constructive formula (51), which applies to all other quintics of class B when (52) does

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* Namely, for use in (25), $r_1 = -\zeta$, $r_2 = -(\gamma^{-2}\delta^3 + 4\gamma\delta)$, $s_1 = \frac{1}{16}(r_1^2 + r_2^2)$, $s_2 = \frac{1}{8} r_1 r_2$. This may be improved by writing $r_2 = -\zeta - 2\gamma\delta$.

not vanish, needs for this class to be replaced by the different constructive formula (54).

24. Critical Case D: $\gamma^2 - v = n^2t^2 = \frac{1}{2}\delta t$, where $n = (\gamma t + \delta)\,t^{-2}$. The special cases thus far discussed have been those in which $v = 0$, and each of them has in some sort been brought under the general constructive formulæ (39) and (40). Those which we have still to consider are such as cannot properly be included under those formulæ. Our object in framing those formulæ was to solve the diophantine problem presented by the fundamental equations (19–21), in which we have to assign admissible rational values to p, t, and v. To solve this problem in its general form we have found it necessary to make use of the relationships embodied in (42) and (43). In the special case now considered, we find that both sides of (43), and both numerator and denominator of the value of l shown in (42), all vanish identically: that is to say,

$$\left.\begin{aligned} \delta^2 + t^2v + 4\gamma(\gamma^2 - v) &= 0, \\ 2\delta t - 4(\gamma^2 - v) \qquad &= 0. \end{aligned}\right\} \tag{55}$$

Since these equations impose two restrictions upon the values to be assigned to γ, δ, t, v, we have only two parameters at our disposal, t and n. To produce a resolvable quintic of this sort, we must therefore assign rational values to t and n, after which we have, by successive substitution in (40) and (41),

$$\left.\begin{aligned} \delta &= 2n^2t, \\ \gamma &= nt - 2n^2, \\ v &= 4n^3(n - t), \\ \varepsilon &= (5t - 4n)(n^2t - 4n^3), \\ \zeta &= (5t - 4n)^2 . 4n^3. \end{aligned}\right\} \tag{56}$$

For example, let $t = -2$, $n = -\frac{1}{2}$; then $\gamma = \frac{1}{2}$, $\delta = -1$, $\varepsilon = 0$, $\zeta = -32$, and the quintic is $y^5 + 5y^3 - 10y^2 - 32 = 0$. By substitution in (23) and (24) we find $s_1 = s_2 = 0$, and from this we find, by (25), $u_1^5 = u_4^5$, $u_2^5 = u_3^5$. This may mean that $u_1 = \omega u_4$, where ω is a fifth-root of unity, but the only arrangements of such fifth-roots compatible with (5–8) result in exhibiting one root of the quintic as a sum of real elements, so that we may say at once that $u_1 = u_4$, $u_2 = u_3$, and these relations will supply the simplest definition of this special class. All the other special cases have been noticed heretofore, and solutions of them were given in the paper of 1885. This case might be solved by (25), but

these quintics require no solution, as they always have a rational root, the value of which is $-4n$, as may be verified by forming the quintic $y^5+10\gamma y^3+10\delta y^2+5\varepsilon y+\zeta=0$ with the values shown in (56). In fact, $n=-\frac{1}{2}(u_1+u_2)$, $\sqrt{(n^2-nt)}=-\frac{1}{2}(u_1-u_2)$, $nt=u_1u_2$, and m, in (44), becomes indeterminate, so that w also is indeterminate. This case may, from the point of view of the general constructive formulæ (39) and (40), be described as that in which we endeavor to assign to q the impossible value -1. One remarkable feature is that v may be negative, although essentially positive in (40). For example, if, as in the example selected, $n=-\frac{1}{2}$ and $t=-2$, we have $v=-\frac{3}{4}$. The constructive formulæ (40) may be paralleled in this case by supposing

$$\left.\begin{aligned} q &= -1, \\ l &= -\gamma = 2n^2 - tn, \\ r &= t^2 \div (2n-t), \\ w^2 &= t^2 \div 4(n^2-nt). \end{aligned}\right\} \tag{57}$$

We learn from this that w^2 is negative when we assign to t a value greater than to n, and of the same sign; and we see also that v is negative whenever w^2 is negative. The value of n in terms of the coefficients, as derived from (56), is

$$n=\frac{(\delta^2\zeta-4\delta\varepsilon^2)(\delta\zeta-\varepsilon^2)+8\gamma^3\delta\zeta^2}{8\gamma(\delta\zeta-\varepsilon^2)^2+16\gamma^4\zeta^2+8\gamma\delta\varepsilon^2\zeta-2\gamma\delta^2\zeta^2}. \tag{58}$$

When we take $t=n$, we have $v=0$, $\gamma=-n^2$, $\delta=2n^3$, $u_1=u_2=u_3=u_4=-n$, that is to say, we have case C as a special form of case D, as well as of case B.

25. Critical Case E: $v=\gamma^2$, $u_4=0$, $\gamma=-\frac{1}{2}u_2u_3$, $\delta=-\frac{1}{2}(u_2^2u_1+u_1^2u_3)$, $\gamma t=tv^{\frac{1}{2}}=\frac{1}{2}(u_2^2u_1-u_1^2u_3)$, and from (10 and (17),

$$\begin{aligned} \varepsilon &= 4\gamma^2-u_1^3u_2-u_3^3u_1=4\gamma^2-u_1^2u_3\left[u_2^2u_1(u_2u_3)^{-1}+(u_2u_3)^2(u_2^2u_1)^{-1}\right] \\ &= 4\gamma^2+(\delta+\gamma t)\left[\tfrac{1}{2}\gamma^{-1}(\delta-\gamma t)-4\gamma^2(\delta-\gamma t)^{-1}\right], \end{aligned} \tag{59}$$

$$\begin{aligned} \zeta &= -20\gamma^2t-u_1^5-u_2^5-u_3^5 \\ &= -20\gamma^2t-\tfrac{1}{4}\gamma^{-2}hk^2+2\gamma h^2k^{-1}-16\gamma^4h^{-2}k, \end{aligned} \tag{60}$$

where $h=u_2^2u_1=-\delta+\gamma t$, $k=u_1^2u_3=-\delta-\gamma t$. It appears from these explicit formulæ that in this case we are at liberty to assign values at will to γ, δ, t, or, what is the same thing, to h, k, t. If, for example, we take $\gamma=\frac{1}{2}$, $\delta=\frac{1}{2}$, $t=3$, we have $h=1$, $k=-2$, $\varepsilon=1$, $\zeta=-\frac{35}{2}$. The expressions for u_1^5, u_2^5, u_3^5, contained in (60) afford an easy means of solution when t is known,

and it is of course unnecessary to extract the fifth-root of more than one of these quantities, owing to the relations existing between γ, h, and k.. An expression for t in terms of the coefficients, as well as a formula showing the necessary relation between the coefficients when $v=\gamma^2$, was given in the former paper. From (59) and (60) we perceive that when $\gamma=0$, either h or k must vanish also; that when $h=0$, either $\gamma=0$ or $k=0$; and that when $k=0$, either $\gamma=0$ or $k=0$; in other words, we are forbidden to make one of these three quantities zero unless we give the same value to a second of them. Again, when $t=0$, $h=k=-\delta$, and the formulæ become simplified. The same may be said if $\delta=0$, when $h=-k$. In the extreme case wherein $\delta=0$, and $h=k=t=0$, we have De Moivre's solvable quintic, which may also be defined by $u_1=u_4=0$; namely, $y^5+10\gamma y^3+20\gamma^2 y+\zeta=0$, where $\gamma=-\frac{1}{2}u_2u_3$ and $\zeta=-u_2^5-u_3^5$. If, on the other hand, besides $u_4=0$, we have either $u_2=0$ or $u_3=0$, we have $\gamma=0$, $t=\infty$, $\gamma t=\pm\delta$, and either $h=0$ or $k=0$ respectively. In the one case, $\delta=-\frac{1}{2}u_1^2u_3$, $\varepsilon=-u_1u_3^3$, $\zeta=-u_1^5-u_3^5$; in the other, $\delta=-\frac{1}{2}u_2^2u_1$, $\varepsilon=-u_1^3u_2$, $\zeta=-u_1^5-u_2^5$, the two cases differing only by an interchange of subscripts. This is Euler's solvable quintic, wherein $\gamma=0$, $16\delta^4-\varepsilon^3+2\delta\varepsilon\zeta=0$. Again, if both u_2 and u_3 vanish as well as u_4, we have the binomial quintic $y^5+\zeta=y^5-u_1^5=0$. We may harmonize this special case E with the general constructive formulæ (39) and (40) by prescribing the following limitations upon the values to be assigned to q, r, w: let g be any rational quantity except 0 or ± 1, and let $q=2g\div(1+g^2)$, let $w=(1-g^2)\div 2g$, and let $r=qwt$; then (39) becomes indeterminate, and l remains at our disposal, so that we may assign to it any rational value except 0. We have thus three parameters, g, l, and t, which are connected with γ, δ, t, by the relations $g\delta=\gamma t$, $2l\delta t=\delta^2+\gamma^2t^2$, $\gamma=ql$, $\delta=(r+t)\,l$. Thus, having $t=3$, $\gamma=\frac{1}{2}$, $\delta=\frac{1}{2}$, we find readily that $l=\frac{5}{6}$ and $g=3$, whence $q=\frac{3}{5}$, $w=-\frac{4}{3}$, $r=-\frac{12}{5}$. We might therefore have begun by assigning $g=3$, $l=\frac{5}{6}$, $t=3$, and thereby finding $\gamma=\frac{1}{2}$, $\delta=\frac{1}{2}$, after which the rest of the quintic would be constructed by means of (59) and (60). It may be observed that the value $t=0$ corresponds to values of u_1, u_2, u_3 in geometrical progression: $u_2^2=u_1u_3$, $h=k$, $m=1$, $w=\infty$, $g=0$, $l=\infty$, $\varepsilon=\frac{1}{2}\gamma^{-1}\delta^2$, $\zeta=16\gamma^4\delta^{-1}-2\gamma\delta+\frac{1}{4}\gamma^{-2}\delta^3$. In the extreme case wherein $t=0$, and $u_1=u_2=u_3$, we have $\gamma=-\frac{1}{2}u_1^2$, $\delta=-u_1^3$, $\varepsilon=-u_1^4$, $\zeta=-3u_1^5$, as for example in $y^5-5y^3-10y^2-5y-3=0$, which has a root $y=3u_1=3$.

26. If in the general formula (39) for l we put $q=0$, $r=\frac{4}{5}$, $t=0$, $w=-\frac{3}{4}$,

we shall have $l=\frac{1}{4}$, and from (40) and (41) we derive $\gamma=0$, $\delta=\frac{1}{5}$, $\varepsilon=0$, $\zeta=\frac{1}{5}$, so that the quintic is

$$y^5+2y^2+\tfrac{1}{5}=0. \tag{61}$$

To produce this example of a resolvable quintic of the form $y^5+10\gamma y^3+10\delta y^2+5\varepsilon y+\zeta=0$, in which $\varepsilon=0$, I have been compelled to reduce two of the four parameters to zero. Whether any method of constructing such quintics can be devised which shall place three parameters at our disposal is a question which the future must determine. When we desire to suppress γ or δ, we have only to put $q=0$ or $r=-t$; to suppress ε and still have three parameters free is a desideratum, which may eventually prove to be an impossibility.

27. The reader will naturally be interested in the inquiry whether the four parameters are presented in their simplest form. It is obvious that any four quantities which are rational functions of these parameters, and of which the parameters are rational functions, could be made to serve as parameters in lieu of q, r, t, and w. To examine this question let us set before us the conditions which have to be satisfied, namely, from (42–44),

$$l=\frac{\delta^2+t^2v+4\gamma(\gamma^2-v)}{2\delta t-4(\gamma^2-v)}=v^{\frac{1}{2}}\,\frac{m^2+1}{m^2-1}, \tag{62}$$

$$l^2=v(1+w^2), \tag{63}$$

$$w=\frac{2m}{m^2-1}, \tag{64}$$

$$m=\frac{u_1^2u_3-u_4^2u_2}{u_2^2u_1-u_3^2u_4}. \tag{65}$$

Considered as functions of the roots or of the elements, t is of weight 1, γ and l of weight 2, δ of weight 3, v of weight 4, w and m having no weight. Our object is to select as parameters four quantities which shall be rational functions of t, γ, δ, v, and w, and of which the latter shall be rational functions. It is necessary to include w in this list, or at any rate w multiplied by δ, t, or v, or by any two, or by all three, since the expression for ε in (40) includes δtvw. It would not improve (63) were we to substitute for w its equivalent in terms of weighted quantities such as δw or tw, nor could any suitable function of w, taking its place, be represented so simply as a function of the elements as w in (64). It is not conceivable that the form in which w is presented can be improved upon, though it might of course be affected by a numerical multiplier.

It will not be forgotten that m is not admissible as a parameter, being usually irrational. The more we examine t the more we shall be convinced that, as representing weight 1, it cannot be replaced as a parameter by anything else than nt, where n is some number, positive or negative, a change which appears to present no advantage. We have therefore to find two parameters which, with t and w, shall take the place of γ, δ, v, and l, in (62) and (63). The first temptation is to assume that the weights of γ, δ, v, and l, are derivable from t, as in $\gamma = ht^2$, $\delta = it^3$, $l = jt^2$, but the only good purpose served by this suggestion is to make it clear that γ, δ, and l must be made simple functions of some quantity such that the employment of (62) shall exhibit such quantity as a rational function of t, w, and two other parameters. As γ and l are of lower weight than δ, it seems desirable to take as such quantity either γ or l. If we assume $l = h\gamma$, and $\delta = (k + ht)\gamma$, we shall derive from (62), upon due substitution,

$$\gamma = \frac{h^2 t^2 w^2 - k^2 (1 + w^2)}{4(h+1)(1 + w^2 - h^2)}, \tag{66}$$

and since $\delta = \gamma (k + ht)$ and $v = h^2\gamma^2 \div (1 + w^2)$, we have here a solution of the difficulty. This is, in fact, the same as (39), provided we write h^{-1} for q and $h^{-1}k$ for r, and multiply both sides of (39) by h^{-1} or q. The use of (66), however, is not to be preferred, since of itself it is rather less simple than (39), while the expressions for δ and v are decidedly less satisfactory. With (66) we cannot even make $\gamma = 0$ without first putting $h = \infty$. We therefore find it necessary to have $\gamma = ql$; and it only remains to consider why we should have $\delta = (r + t)\,l$ instead of, say, $\delta = gl$ or $\delta = (g + 1)\,tl$. The latter is not admissible because in fact δ often vanishes while t does not, and *vice versa*. The reason why it is preferable to write $\delta = (r + t)\,l$, where r is a parameter of weight 1, is that by this arrangement we are enabled to reduce to its simplest form the construction of sets of conjugate resolvable quintics.

28. Quintics may be called conjugate when they have the same resolvent t and the same value of v corresponding to t. It appears that they must also have the same value of γ. The existence of one quintic conjugate to any given resolvable quintic was discovered and pointed out by me some years ago, when I showed how its coefficients could be determined from those of the given quintic, knowing t and v, by the aid of a quadratic equation. The method presented in

the present paper for constructing at will resolvable quintics by the aid of four rational parameters enables us also to construct their conjugates at once. The second quintic, that is to say, the conjugate formerly discovered, may now be constructed by simply changing the sign of the parameter w. A third may be constructed by changing the sign of the parameter r; and a fourth by changing the signs of both w and r. Since v contains w and r only as w^2 and r^2, such changes of sign leave the value of v unaltered. Thus, for example, if we have $t=4$, $q=0$, $r=\pm 2$, $w=\pm 1$, we derive from (39) and (40) these four conjugate quintics, for each of which $t=4$ and $v=2$:

$$\begin{aligned}
y^5 - 120y^2 - 410y - 1200 &= 0,\\
y^5 - 120y^2 + 470y - 496 &= 0,\\
y^5 - 40y^2 - 90y - 240 &= 0,\\
y^5 - 40y^2 + 150y - 48 &= 0.
\end{aligned}$$

In the special case E, wherein $v=\gamma^2$, and the three parameters are γ, δ, and t, there is no limit to the possible number of conjugates having identical values for t and v. In case D there appear to be no conjugates, and in cases B and C, in which t is indeterminate, the question does not arise. In case A, wherein $v=0$ and $t=\infty$, quintics having the same value for s may be regarded as conjugate. If we attempt to construct a general resolvable quintic which shall have no conjugate, by putting $r=0$ and $w=0$ in (39), we merely get $l=0$, whence $\gamma=0$, $\delta=0$, $\varepsilon=0$.

29. The resolvent (34), and therefore all other possible resolvents, can be expressed rationally in terms of the four parameters, and when so expressed becomes identically zero. As a simple illustration, let us take $q=0$, $r=-t$, so that $\gamma=0$, $\delta=0$, and the quintic (2) is reduced to the trinomial form

$$y^5 + 5\varepsilon y + \zeta = 0. \tag{67}$$

In this case my general resolvent becomes

$$\varepsilon t^6 - \zeta t^5 - 20\varepsilon^2 t^2 - 4\varepsilon\zeta t - \zeta^2 = 0, \tag{68}$$

which will doubtless be found the simplest resolvent possible for this trinomial form. By (40) and (47),

$$\left.\begin{aligned}
\varepsilon &= v(4w+3),\\
\zeta &= tv(4w-22),
\end{aligned}\right\} \tag{69}$$

where, by (40), $v = l^2 \div (1 + w^2)$, and, by (39), $l = \frac{1}{4} t^2$. Substituting for ε and ζ in (68) their values from (69), and dividing throughout by $t^2 v$, we have

$$t^4 (4w + 3) - t^4 (4w - 22) - 20v (4w + 3)^2 \\ - 4v (4w + 3)(4w - 22) - v (4w - 22)^2 = 0, \quad (70)$$

or

$$25t^4 - 400v (1 + w^2) = 0,$$

which is an identity. Students of the quintic who prefer to deal only with the trinomial form (67) will find that the derivation of (68) from (70) by the aid of (69) covers their ground pretty well. Those, however, who suppose that the reduction of the general quintic to the form (67), by means of the Bring-Jerrard transformation, is a physically available process will do well to make a personal experiment by performing that transformation in some numerical case. It will be found far more difficult than the use of the resolvent (34, 37).

30. The expressions (69) just noted for the construction of resolvable quintics of the form $y^5 + 5\varepsilon y + \zeta = 0$ are, though special cases of (39) and (40), equivalent in substance to certain formulæ devised by Professor G. P. Young, which were published, in different forms, in the same number of the American Journal of Mathematics (VII, 170, 178) by that writer under his own name and that of Mr. J. C. Glashan. As I have stated that the use of the quantity t was original with myself, it must be explained that Professor Young employed on this occasion a symbol equivalent to t confined to the trinomial only, a symbol moreover which he introduced late in the discussion to represent the fourth-root of a fraction; and that Mr. Glashan's $-2k$, which happened to correspond to t for this trinomial, had a widely different meaning in its general definition, as we shall see when we come to consider Mr. Glashan's work further on. The same remark applies to his m, which for this trinomial is equivalent to $-w$. I had previously made extensive use of t, as here defined for the general quintic, in vol. VI of the same journal, except that it was there taken with the opposite sign. Concerning the trinomial form in question Mr. Glashan said that "the solvable quintic assumes the form

$$x^5 + 5\left(\frac{3 - 4m}{1 + m^2}\right) k^4 x + 4\left(\frac{11 + 2m}{1 + m^2}\right) k^5 = 0; \quad (71)$$

a form communicated to the present writer [Glashan] by Professor G. P. Young of Toronto University in May, 1883." I dwell with some particularity upon the

history of the formulæ available for constructing resolvable trinomials of the form $y^5 + 5\varepsilon y + \zeta = 0$, because such formulæ are the only ones known hitherto by which resolvable quintics can actually be constructed, otherwise than with rational roots or elements. The two papers mentioned appeared late in 1884 or early in 1885. Later in the same year, 1885, the following formula for the resolvable trinomial was published in the Acta Mathematica* by Runge:

$$x^5 + \frac{5\mu^4(4\lambda+3)}{\lambda^2+1}x + \frac{4\mu^5(2\lambda+1)(4\lambda+3)}{\lambda^2+1} = 0. \tag{72}$$

Here $\mu = -\frac{1}{2}t$, and $\lambda = \dfrac{4-3w}{4w+3}$. In 1890 the following was published at Moscow† by Bugaieff and Lachtine:

$$(\lambda x)^5 + \frac{(\mu-1)(\mu-11)}{4(\mu^2+4)}(\lambda x) + \frac{\mu-11}{2(\mu^2+4)} = 0. \tag{73}$$

Here $\lambda = t^{-1}$, and $\mu = 2\dfrac{2+11w}{2w-11}$.

31. With a single exception, the situation up to the present time‡ is this, that except for the trinomial case $y^5 + 5\varepsilon y + \zeta = 0$, no method for constructing resolvable quintics has ever been attempted, apart, of course, from the giving of rational values to the elements or to the roots. The exception consists in certain formulæ, or fragment of a paper, which appeared in 1884 or 1885 under the name of Mr. Glashan as already mentioned. I learn from Mr. Glashan that early in 1883 Professor Young deposited with him the trinomial formula (71) under seal, stating that there was such a formula enclosed; that he thereupon endeavored to produce a corresponding formula to compare with Professor Young's when it should be opened, and communicated his results, without proof, to Professor Young, who subsequently caused his somewhat hasty sketch to be published in the American Journal without his knowledge. Meanwhile, his intricate preliminary work had been mislaid or destroyed. It is thus explained how the paper originated, how it came to appear without demonstration while yet including errors of detail, and why the author made no subsequent correc-

* Cited in the Fortschritte as in vol. VIII; in Weber's Algebra as in vol. VII. The trinomial form given by Weber, I, 626, is erroneous.

† Cited in Fortschritte, XXII, 114.

‡ Perhaps I should mention that I communicated the construction-method of (39) and (40) to Professor E. H. Moore in September, 1894, by letter.

tion. Considering the state of the theory at the time when this remarkable fragment appeared, it must, notwithstanding its imperfections, be recognized as a bold and able advance upon untrodden ground. Had Mr. Glashan had an opportunity of correcting his formulæ before they were published and of supplying the demonstration, they would, although not meant to cover the whole field of resolvable quintics, but expressly introduced as applying only to a group of such quintics,* have constituted a notable advance upon the only formulæ of the sort known up to the present time, those namely for constructing the trinomial $y^5 + 5\varepsilon y + \zeta = 0$. In particular, Mr. Glashan's formulæ, if they had been corrected, would have covered, among other resolvable quintics, the whole of the class of quadrinomials of the form $y^5 + 10\gamma y^3 + 5\varepsilon y + \zeta = 0$, a form which includes as special cases both the trinomial already mentioned and the trinomial $y^5 + 10\gamma y^3 + \zeta = 0$. It has been a matter of some difficulty for me to work out the points of connection between what Mr. Glashan's formulæ would have been, if set right, and the general system now presented, but I have succeeded in doing so, and shall now present Mr. Glashan's system correctly, though in my own language.

32. If our general parameters, q, r, t, w, are made rational functions of four other parameters, g, k, m, n, which at the same time are rational functions of q, r, t, w, the new parameters must take the place of the old without lessening the generality of the system, and the group covered by them is the whole mass of resolvable quintics. If, on the other hand, we make q, r, t, w, rational functions of g, k, m, n, while the latter are not all rational functions of the former, we may assign rational values at will to g, k, m, n, and thereby always produce resolvable quintics, but our field is now restricted to a smaller group, outside of which are to be found all resolvable quintics produced by assigning rational values to q, r, t, w, which do not correspond to rational values of g, k, m, n. Mr. Glashan's parameters were g, k, m, n, of which q, r, t, w, are rational functions, but to obtain which from q, r, t, w, requires (except when $r = -t$) the intervention of a biquadratic equation. The relations between

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* His words were: "the coefficients must be so related that *if* $p_2 = n\theta k^2$, $p_3 = \alpha\theta^2 k^3$, and $p_4 = \beta\theta^2 k^4$, *then* must $p_5 = 2(1+n)\gamma\theta^2 k^5$," etc., the coefficients being p_2 for our γ, p_3 for our δ, etc. I have italicized two words.

the general parameters q, r, t, w, underlying all resolvable quintics, and Mr. Glashan's parameters g, k, m, n, pertaining to a group only, are:

$$\left.\begin{aligned} t &= \theta k(\alpha\lambda - 2), \\ q &= n\mu, \\ r &= \theta k(\alpha\mu - \alpha\lambda + 2), \\ w &= 2\,\frac{\alpha\eta(m-g)-m}{\alpha\eta(\alpha\lambda-2)+2}, \end{aligned}\right\} \tag{74}$$

where $\eta = (1+n)(1+m^2)$, $\theta = 1 + n - n^2\eta$, $\lambda = 1 + m^2 - g^2 - ng^2$, $\alpha = 4g(m - g - ng) \div (\lambda^2 - 1 - m^2)$, and $\mu = [2\alpha\eta(\alpha\lambda - 2) + 4] \div [\alpha^2\eta + (\alpha\lambda - 2)^2(1+n) - 4n]$. It will be seen that t, q, r, w, are all rational when g, k, m, n, are rational. Substituting these values in (39) we have $l = \mu^{-1}\theta k^2$, and it may also be proved by due substitution that $\mu^2(1+w^2) = 1 + m^2$. Hence, from (40), $\gamma = n\theta k^2$, $\delta = \alpha\theta^2k^3$, $v = \theta^2k^4 \div (1+m^2)$. Then ε and ζ may be determined by (40) and (41). For example, let the group parameters be $g = -1$, $k = \frac{1}{4}$, $m = 2$, $n = 1$; then $\eta = 10$, $\theta = -8$, $\lambda = 3$, $\alpha = -4$, $\mu = \frac{281}{137}$, and by (74) the general parameters are $t = 28$, $q = \frac{281}{137}$, $r = -\frac{1588}{137}$, $w = -\frac{122}{281}$. We find that $\gamma = -\frac{1}{2}$, $\delta = -4$, $v = \frac{1}{20}$, $\varepsilon = 34$. It has been said that to ascertain the values of g, k, m, n, from those of t, v, and the coefficients, it is necessary in general to find one of the roots of a biquadratic equation: in the present example the biquadratic must have a rational root. The quantities t, v, q, r, w, l, p, r_1, r_2^2, s_1, s_2^2, all belong to one family of functions, conjugate to which are five other like families, since each of these quantities is a six-valued function of the roots of the quintic. When we find, by means of the resolvent, a rational value of t, we are enabled, by means of the formulæ herein given, to obtain the corresponding values of the other functions comprised in the same family, and we have, for present purposes, nothing to do with the other five sets of values. The various quantities contained in one family with the given value of t are each and all rational functions of t and of each other, and they have therefore been spoken of herein as "rational." It is unnecessary to remark that the relations between them hold good, and that they are all still rational functions, each of each other, when the value of t is not rational, and when therefore all other quantities of the same family are in general irrational. The quantities g, k, m, n, discussed by Mr. Glashan, belong to one subdivision of this family of functions.

33. Thus far we have dealt almost altogether with the shorter form (2) of the general quintic, for which my resolvent is the quantity t. The broader resolvent in τ was exhibited in (37) as the covariant sextic $\mathrm{AL} - 25\mathrm{C}^2 = 0$. It will be remembered that ϕ is the same function of x, in the longer form (1) of the general quintic, as it is of y in the shorter form (2), but that in (1) we have $\psi = t\phi - ba^{-1}\phi = \sigma - ba^{-1}\phi$, and $\tau = \psi\phi^{-1} = t - ba^{-1}$, just as $x = y - ba^{-1}$. It has also been remarked that for the longer form of the general quintic the quantity ba^{-1} may be regarded in a sense as a fifth parameter, though it is much broader as a function of the roots than the four parameters, having only one value instead of six. Let us now examine the form of the six values of τ in detail, τ being the leading parameter of the general quintic (1), reduced to t when $ba^{-1} = 0$, as in the shorter quintic (2).

34. In paragraph 13 we had the symbol C—cycle or circle—so defined that $\mathrm{C}f(x_1, x_2, x_3, x_4, x_5) = f(x_1, x_2, x_3, x_4, x_5) + f(x_2, x_3, x_4, x_5, x_1) + f(x_3, x_4, x_5, x_1, x_2) + f(x_4, x_5, x_1, x_2, x_3) + f(x_5, x_1, x_2, x_3, x_4)$; that is to say, C represented the sum of five similar functions of the roots, each function differing from the one preceding by a change in the subscripts, from 1 to 2, and so on, in the order 12345. Let us hereafter write C_1 for C as heretofore used in this sense. For the family of six-valued functions of the roots with which we are concerned, there are five other possible sequences, which may be distinguished by assigning different subscripts to C,* thus:

$$\left.\begin{array}{ll} \mathrm{C}_1 : 12345 & \mathrm{C}_4 : 12453 \\ \mathrm{C}_2 : 12534 & \mathrm{C}_5 : 13425 \\ \mathrm{C}_3 : 14235 & \mathrm{C}_6 : 13254 \end{array}\right\} \tag{75}$$

We have had $\phi_1 = \mathrm{C}_1 x_1 (x_2 - x_3)$, and $\psi_1 = \mathrm{C}_1 x_1 (x_2 - x_3) x_5$. Similarly, $\phi_2 = \mathrm{C}_2 x_1 (x_2 - x_5)$, $\psi_2 = \mathrm{C}_2 x_1 (x_2 - x_5) x_4$; $\phi_3 = \mathrm{C}_3 x_1 (x_4 - x_2)$, $\psi_3 = \mathrm{C}_3 x_1 (x_4 - x_2) x_5$; and so on. It will be observed that each ϕ consists of five positive products, each of two adjacent roots, according to the sequence employed, plus five negative products, each of two non-adjacent roots. Similarly, each ψ consists of five positive products, each of three adjacent roots, plus five negative products, each of two adjacent and one non-adjacent roots. Then the six values of τ are $\tau_1 = \psi_1 \phi_1^{-1}$; $\tau_2 = \psi_2 \phi_2^{-2}$; and so on.

* The subscripts were arranged differently in the paper of 1885, following Cayley. It is necessary for symmetry to make certain changes.

35. I say now that

$$c_6\,(\psi_1\phi_3-\psi_3\phi_1)(\psi_2\phi_4-\psi_4\phi_2)(\psi_5\phi_6-\psi_6\phi_5)=0\,. \tag{76}$$

It will shortly be shown that it is an alternating function, that is to say, one which has the same value, but with the opposite sign, whenever any two roots are transposed one for the other. As such, being of the fifteenth degree in the roots, it must be composed of two factors, one namely of the tenth degree, consisting of the product of the differences of the roots, $(x_1-x_2)(x_1-x_3)\,\ldots\,(x_4-x_5)$, which changes sign when a transposition occurs, the other a symmetrical function of the fifth degree, upon which a transposition produces no effect. If, having this in mind, and expecting the product to consist of a large number of terms, we attempt to evaluate (76) we are met nevertheless by a zero coefficient for the first term we deal with, and again for the next term, whatever it may be, and so on, until we cannot but be convinced that the symmetrical factor must vanish, if it exists at all even as a phantom. To evaluate the myriads of terms contained in the product would be impossible. It is, however, feasible to reduce greatly the number of terms which it is necessary to examine. Every term found non-existent proves the non-existence of every other term of the same form with different subscripts, since transposition leaves the value unchanged unless in sign. Each of the three factors of (76) consists of terms in which no one root enters in a power above the second, so that no term of the product can contain any power of one root above the sixth. The three factors of (76) contain 24 terms each, which when examined show that the product can contain no term of three letters, and that no term can, as to its literal part, be a perfect cube. Each factor is of the form $\psi_m\phi_n-\psi_n\phi_m$, and every term of $\psi_m\phi_n$ is represented in $\psi_n\phi_m$ by a complementary term affected by the same coefficient, the literal parts being such that the product of the two is $x_1^6x_2^6x_3^6x_4^6x_5^6$; and it follows that when any term in the product is found to vanish, the complementary term must likewise vanish. For example, $x_1^5x_2^4x_3^4x_4^2$ is complementary to $x_1x_2^2x_3^2x_4^4x_5^6$, equivalent as a form to $x_1^6x_2^4x_3^2x_4^2x_5$. For the same reason there can be no term $x_1^3x_2^3x_3^3x_4^3x_5^3$, which is besides to be set aside as a cube. It thus results that the number of terms in the product which, to make sure that every term vanishes it is necessary to examine, and which I have examined, is reduced to sixteen. It is unnecessary to present the details, but to assist any one who may wish to look into the matter I illustrate the process of evaluation by an example.

36. Let us take $x_1^6x_2^5x_3^3x_4$ as the term in (76) whose coefficient is to be examined. Now (76) consists of five grand terms, the first of which is $(\psi_1\phi_3 - \psi_3\phi_1)$ $(\psi_2\phi_4 - \psi_4\phi_2)(\psi_5\phi_6 - \psi_6\phi_5)$, and the others are derived from this by successive repetition of the cyclic substitution $x_1x_3x_2x_5x_4$. The coefficient of $x_1^6x_2^5x_3^3x_4$ in the second grand term is therefore the same as that of $x_4^6x_3^5x_1^3x_5$ in the first; in the third is the same as that of $x_5^6x_1^5x_4^3x_2$ in the first; in the fourth the same as that of $x_2^6x_4^5x_5^3x_3$, and in the fifth as that of $x_3^6x_5^5x_2^3x_1$. The sum of the coefficients of these expressions in the first grand term is therefore the coefficient desired from the whole of (76) for $x_1^6x_2^5x_3^3x_4$. The three factors of which the first grand term is composed consist each of 24 terms, of which I quote the following (all to be multiplied by 2) as bearing on the present example: in the first factor, $-x_1^2x_3x_4^2$, $x_1x_3^2x_4^2$, $-x_2^2x_3^2x_5$, $-x_2^2x_4^2x_5$, $2x_1^2x_2x_3x_4$; in the second factor, $-x_1^2x_2^2x_3$, $-x_1x_3^2x_5^2$, $x_2x_3^2x_5^2$, $-x_3^2x_4^2x_5$, $2x_2^2x_3x_4x_5$; and in the third factor, $x_1^2x_2^2x_3$, $x_1^2x_3x_4^2$, $-x_1x_3^2x_4^2$, $-x_1x_3^2x_5^2$, $-x_2^2x_4^2x_5$, $x_2x_3^2x_5^2$. We find that $2x_1^2x_2x_3x_4(-x_1^2x_2^2x_3)x_1^2x_2^2x_3 = -2x_1^6x_2^5x_3^3x_4$; that $-x_1^2x_3x_4^2(-x_3^2x_4^2x_5)(-x_1x_3^2x_4^2) = -x_4^6x_3^5x_1^3x_5$, and $x_1x_3^2x_4^2(-x_3^2x_4^2x_5)(x_1^2x_3x_4^2)$ $= -x_4^6x_3^5x_1^3x_5$; that there is no term $x_5^6x_1^5x_4^3x_2$; that $-x_2^2x_4^2x_5(2x_2^2x_3x_4x_5)(-x_2^2x_4^2x_5)$ $= 2x_2^6x_4^5x_5^3x_3$; and that $-x_2^2x_3^2x_5(-x_1x_3^2x_5^2)x_2x_3^2x_5^2 = x_3^6x_5^5x_2^3x_1$, and $-x_2^2x_3^2x_5 \,.\, x_2x_3^2x_5^2$ $(-x_1x_3^2x_5^2) = x_3^6x_5^5x_2^3x_1$. The sum of the coefficients is zero, and there is therefore in (76) no term $x_1^6x_2^5x_3^3x_4$.

37. It remains to be shown that (76) merely changes sign when any two roots are transposed. Let us consider first the first grand term $(\psi_1\phi_3 - \psi_3\phi_1)$ $(\psi_2\phi_4 - \psi_4\phi_2)(\psi_5\phi_6 - \psi_6\phi_5)$, and let us begin by excluding the root x_4 from transposition. In the following schedule we have the effects produced upon the several functions ψ by the transpositions noted at the left, and it is to be observed that the same effects, as to both signs and subscripts, are produced upon the corresponding functions ϕ.

$$\left.\begin{array}{c|cc|cc|cc}
 & \psi_1 & \psi_3 & \psi_2 & \psi_4 & \psi_5 & \psi_6 \\
\hline
(x_1x_2) & -\psi_3 & -\psi_1 & -\psi_6 & -\psi_5 & -\psi_4 & -\psi_2 \\
(x_1x_3) & -\psi_5 & -\psi_6 & -\psi_4 & -\psi_2 & -\psi_1 & -\psi_3 \\
(x_1x_5) & -\psi_4 & -\psi_2 & -\psi_3 & -\psi_1 & -\psi_6 & -\psi_5
\end{array}\right\} \qquad (77)$$

Each of the transpositions (x_1x_2), (x_1x_3), (x_1x_5), therefore, affects the first grand term only by changing its sign, and as all other transpositions of x_1, x_2, x_3, x_5, may be transformed into an odd number of the transpositions named, the same

is true of all transpositions not affecting x_4. Since the other grand terms are derived from the first by successive cyclic substitutions according to the sequence $x_1x_3x_2x_5x_4$, any transposition affects every grand term only by a change of sign, except transpositions including x_4 for the first grand term, x_1 for the second, x_3 for the third, x_2 for the fourth, and x_5 for the fifth. Any single transposition, say (x_1x_4), affects three grand terms only by a change of sign, and we shall now see that, as regards the other two grand terms, it merely transforms each into the other, again with a change of sign. For (x_1x_4) changes ψ_1 into ψ_2, ψ_2 into ψ_1, ψ_3 into ψ_5, ψ_4 into ψ_6, ψ_5 into ψ_3, and ψ_6 into ψ_4, in each case with a change of sign, and the same effects are produced upon the functions ϕ. Hence (x_1x_4) transforms the first grand term $(\psi_1\phi_3-\psi_3\phi_1)(\psi_2\phi_4-\psi_4\phi_2)(\psi_5\phi_6-\psi_6\phi_5)$ into the second grand term with its sign changed, namely, $-(\psi_2\phi_5-\psi_5\phi_2)(\psi_4\phi_3-\psi_3\phi_4)(\psi_1\phi_6-\psi_6\phi_1)$, and *vice versa*, so that (x_1x_4) alters (76) only by changing its sign. The cyclic substitution $(x_1x_3x_2x_5x_4)$ leaves (76) unaltered, so that successively we have, after (x_4x_1) the transpositions (x_1x_3), (x_3x_2), (x_2x_5), (x_5x_4), all affecting (76) only by a change of sign; and since all possible transpositions are composed of odd numbers of the transpositions named, any single transposition can affect (76) only by a change of sign. It should be remarked that the sequence 13254 in the roots is equivalent to 12435 in the subscripts of ϕ and ψ.

38. By expanding and rearranging the terms of (76) we have, after dividing throughout by 2,

$$\begin{aligned}
&\psi_1\psi_3\psi_6\phi_2\phi_4\phi_5-\psi_1\psi_2\psi_6\phi_3\phi_4\phi_5\\
+&\psi_3\psi_2\psi_6\phi_1\phi_4\phi_5-\psi_3\psi_5\psi_6\phi_1\phi_2\phi_4\\
+&\psi_2\psi_5\psi_6\phi_1\phi_3\phi_4-\psi_2\psi_4\psi_6\phi_1\phi_3\phi_5\\
+&\psi_5\psi_4\psi_6\phi_1\phi_2\phi_3-\psi_5\psi_1\psi_6\phi_2\phi_3\phi_4\\
+&\psi_4\psi_1\psi_6\phi_2\phi_3\phi_5-\psi_4\psi_3\psi_6\phi_1\phi_2\phi_5\\
+&\psi_1\psi_2\psi_4\phi_3\phi_5\phi_6-\psi_1\psi_3\psi_4\phi_2\phi_5\phi_6\\
+&\psi_3\psi_5\psi_1\phi_2\phi_4\phi_6-\psi_3\psi_2\psi_1\phi_4\phi_5\phi_6\\
+&\psi_2\psi_4\psi_3\phi_1\phi_5\phi_6-\psi_2\psi_5\psi_3\phi_1\phi_4\phi_6\\
+&\psi_5\psi_1\psi_2\phi_3\phi_4\phi_6-\psi_5\psi_4\psi_2\phi_1\phi_3\phi_6\\
+&\psi_4\psi_3\psi_5\phi_1\phi_2\phi_6-\psi_4\psi_1\psi_5\phi_2\phi_3\phi_6=0. \qquad (78)
\end{aligned}$$

Substituting $\tau\phi$ for ψ, and dividing throughout by $\phi_1\phi_2\phi_3\phi_4\phi_5\phi_6$, we perceive

that (78) can be collected at once into this form (also an alternating function of the roots):

$$c_6\tau_1(\tau_3-\tau_2)\,.\,\tau_6-c_6\tau_1(\tau_3-\tau_2)\,\tau_4=0\,, \tag{79}$$

where the sequence 13254 applies to the subscripts of τ. Therefore,

$$\tau_6=\frac{c_6x_1(x_3-x_2)\,x_4}{c_6x_1(x_3-x_2)}=\frac{c_6\tau_1(\tau_3-\tau_2)\,\tau_4}{c_6\tau_1(\tau_3-\tau_2)}\,. \tag{80}$$

This means that τ_6 is the same function of the other roots of my resolvent (37) as it is of the roots of the quintic. In other words, this remarkable theorem is true, that if the sextic resolvent in τ have a rational root τ_6, and if the sextic be reduced to a quintic by dividing it by $\tau-\tau_6$, the resolvent of the new quintic will itself have the same root τ_6.*

39. There are therefore framed implicitly an apparently infinite number of resolvable quintics for each one which we may frame directly by means of the method of (39) and (40), the new series being produced one from another by the formation and, so to speak, decapitation of resolvent sextics of the form (37). As an example, we may put $q=0$, $r=-2$, $t=2$, $w=3$, so that from (39) and (40), or from (69), we have $l=1$, $v=\frac{1}{10}$, $\varepsilon=\frac{3}{2}$, $\zeta=-2$, so that the quintic constructed is

$$x^5+\tfrac{15}{2}\,x-2=0\,. \tag{81}$$

Employing the general resolvent (37), or preferably the reduced form (68) which it assumes in this trinomial case, the resolvent for (81) is, after clearing of fractions,

$$3\tau^6+4\tau^5-90\tau^2+24\tau-8=0\,. \tag{82}$$

Dividing this by $\tau-2$, and writing z for τ, we have the produced quintic

$$3z^5+10z^4+20z^3+40z^2-10z+4=0\,. \tag{83}$$

The necessary covariants of this, for developing the sextic resolvent according to (37), are

$$\left.\begin{aligned}
&\mathrm{A}=3\tau^5+10\tau^4+20\tau^3+40\tau^2-10\tau+4\,,\\
&\mathrm{L}=6712\tau-11184\,,\\
&\mathrm{C}=-28\tau^3+40\tau^2-36\tau-88\,.
\end{aligned}\right\} \tag{84}$$

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* As a fact ascertained from a sufficient number of numerical examples, but as yet devoid of proof, this theorem was laid before this Society by the writer on taking his seat as President in December, 1890; and was also, shortly afterwards, communicated in a private letter to Professor Cayley.

The resolvent (37) being $\text{AL} - 25\text{C}^2 = 0$, we see that the work can be somewhat reduced by using $\frac{1}{4}\text{C}$ and $\frac{1}{2}\text{C}$ and $\frac{1}{8}\text{L}$, and we have this resolvent for the produced quintic (83):

$$67\tau^6 + 11196\tau^5 - 8500\tau^4 - 800\tau^3 - 46360\tau^2 - 2464\tau - 29792 = 0. \tag{85}$$

According to (80), this also has $\tau = 2$, and dividing by $\tau - 2$, as before, and writing z_2 for τ, we have this second produced quintic:

$$67z_2^5 + 11330z_2^4 + 14160z_2^3 + 27520z_2^2 + 8680z_2^2 + 14896 = 0. \tag{86}$$

The process may be continued indefinitely, but it becomes more difficult. The first produced quintic (83) might have been framed as an original quintic by assuming the following values for the parameters, besides $ba^{-1} = \frac{2}{3}$: $q = \frac{53}{51}$, $r = \frac{16}{153}$, $t = \frac{8}{3}$, $w = \frac{9}{53}$. The value of v is $\frac{2}{45}$, as may be verified by using the covariants (84) in either of the formulæ (38).

40. The sextic (85) is a function of the coefficients of the quintic (83), supplying seven equations from which it would appear that the five fundamental coefficients of (83) may be obtained in terms of, and as rational functions of, the coefficients of the sextic. We might therefore work backwards, starting from the quintic (86), and multiplying it by $z_2 - 2$ to produce the sextic (85), then using the coefficients of the latter to determine those of the quintic (83). Similarly, from (83), knowing its resolvent 2, we could produce the sextic (82), from which the coefficients of the quintic (81) could be obtained by the solution of seven equations. In like manner it would seem that we could work backwards from our original quintic (81), multiplying it by $x - 2$ to produce a sextic, from which the coefficients of another and, so to speak, prior quintic could be ascertained, the chain of resolvable quintics thus produced one from the other extending in both directions without apparent limit.

41. The property just discussed, possessed by the sextic (37) in τ, cannot reasonably be looked for in connection with any other resolvent. It depends upon the relation (80) by which the rational resolvent τ_6 is the same function of the other roots of the sextic as it is of the roots of the original quintic, and this can only happen when, as is the case with τ, the function is of weight 1. It is not to be presumed that any other resolvent function of weight 1 exists which is possessed of this peculiar property.

42. I add a suggestion for the benefit of those who may have occasion to employ (34) or (37) to determine whether or not a given numerical quintic is resolvable. The regular course would be to determine the numerical values of the coefficients of the sextic, and then to examine the sextic to see whether it has a rational root. Referring, for example, to the quintic (83), the regular course would be to frame the resolvent (85) and to examine it for a rational root. It is however usually easier to find the values (see paragraph 4) of $\phi(\tau) = \text{AL} - 25\text{C}^2$, for $\tau = -1$, $\tau = 0$, $\tau = 1$, etc., by reference to the known values of A, L, and C, as for example in (84), without actually determining the numerical values of the coefficients of the sextic. In (84), what we have to do, after dividing L by 8 and C by -4, is to find a value of τ which shall reduce to zero $\phi(\tau) = (3\tau^5 + 10\tau^4 + 20\tau^3 + 40\tau^2 - 10\tau + 4)(839\tau - 1398) - 50(7\tau^3 - 10\tau^2 + 9\tau + 22)^2$. The first of these two terms must in that event be positive, and must be divisible by 50, so that τ must be even, and we might thus be led speedily to $\tau = 2$. Proceeding regularly, however, we find $\phi(-1) = -3.30839$; $\phi(0) = -32.19.49$; $\phi(1) = -27.17.167$. Among the divisors there is no ascending sequence with the common difference 67, which is the coefficient of τ^6, but we perceive $-3, -2, -1$, also $1, 2, 3$. We next find $\phi(2) = 0$.

Postscript, *February* 7, 1898.—Critical Case F: $\gamma = -(\delta^2 + t^2 v) \div 2\delta t$. This case is normal as regards solution, but if we attempt to produce a resolvable quintic of this form we find l indeterminate, so that special measures become necessary. In fact, if we substitute $-2\gamma\delta t$ for $\delta^2 + t^2 v$ in (42) we have at once $l = -\gamma$, whence $q = -1$, which causes the denominator of the constructive expression for l in (39) to vanish. It is therefore necessary that the numerator, $t^2 w^2 - r^2 - r^2 w^2$, shall vanish also. To effect this we must assign a rational value to the usually irrational quantity m in the equation $w = 2m \div (m^2 - 1)$, and we must also limit r to the value $r = 2mt \div (m^2 + 1)$. We have therefore three available rational parameters, besides $q = -1$, namely, t, m, γ. Then $l = -\gamma$, $\delta = -(r + t)\gamma$, $v = \gamma^2 \div (1 + w^2)$, and the values of ε and ζ follow from (40) and (41). In this case v is a perfect square and $v^{\frac{1}{2}}$, r_2, and s_2 are rational. For example, let $t = q = -1$, $m = -2$, and $\gamma = 5$; then $w = -\frac{4}{3}$, $r = \frac{4}{5}$, $\delta = 1$, $v = 9$, $\varepsilon = 0$, $\zeta = -22$, and the quintic is $y^5 + 50y^3 + 10y^2 - 22 = 0$. Since this paper was read I have devised many resolvable numerical examples involving the suppression of ε, but my impression, conveyed in the text, that a general method for suppressing ε is probably impracticable, is not weakened.